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THE SCHOLAR'S ALGEBRA

AN INTRODUCTORY WORK ON ALGEBRA

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BY

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Oxford

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PREFACE.

THE present work professes to be an introductory work on Algebra. The reader will not therefore look for anything like an exhaustive treatise. My aim has been to enable the Scholar, who has been well taught in Arithmetic, to take up the study of Algebra, and to understand from the first what it is, what it aims at, and what are the chief forms of its utility. I have thought that some first books on Algebra have too much the air of abridgments of larger works, and that it would be an advantage to approach the subject with more freedom, with a more natural arrangement, and with greater care not to lose sight prematurely of the subject-matter in the complexities of analytical exercises. If I have at all succeeded in my aim, which must be left to the judgment of others. the young mathematician will start with clear views, of which he is sure to find the benefit in his after-course.

I hope too that I may have provided a book that will meet the wants of those physical students who require to know something of Algebra, but whose curriculum does not include the higher analysis.

I would advise both these classes of readers to master the methods of computation which they will find in the book and to acquire at an early date a facility in the use of Logarithms. They will find the four-figure table of Logarithms, given on page 216, of more practical utility than they might suppose.

I have remarked in the Universities a growing disposition to compel the student of the Higher Mathematics to interpret his results numerically. There can be no better guarantee that he understands what he is about.

Besides providing a language adapted for general reasoning about numbers, and giving additional powers to the computer, Algebra has a very suggestive side,—raising problems in regard to the extension of meaning of the negative sign and of symbols generally, which are interesting in themselves, and have important analogies in language and other fields of thought. On these subjects I have been obliged to limit myself to a few hints.

Answers to the numerous Exercises will be found at the end of the book, and also an Index of subjects.

LEWIS HENSLEY.

HITCHIN,
October, 1875.

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ERRATA.

Page 38, Exercise (17), *for* -64 *read* $+64$.
„ 56, „ (15), *dele* $-4x$.
„ 59, „ (14), *for* $2-3a$ *read* $x-3a$.
„ 144, „ (1), *for* $a=1$ *read* $a=2$.
„ „ „ (3), *for* $a=-5$ *read* $a=-4$.
„ 197, lines 9, 15, 20, *for* ma^2 *read* m^2a.
„ 204, line 7, *for* less *read* nearer.
„ 207, lines 3, 4, *for* $'ma^2$ *read* m^2a.

ALGEBRA.

PART I.

THE SYMBOLS, SIGNS, AND ELEMENTARY RULES.

CHAPTER I.

INTRODUCTORY.

(1) Relation of Algebra to Arithmetic.

1. ALGEBRA, like Arithmetic, treats of numbers, but it aims at expressing their relations with greater generality.

2. In Arithmetic we can generalize simple rules and propositions by employing words, as in saying, 'Twice the double of a number is four times that number,' and, 'The sum of two numbers added to their difference is equal to twice the greater.' But just as we could not get far in Arithmetic without figures, so neither can we get far in general reasoning about numbers without some simpler mode of expression than words. Algebra meets this want by introducing, as supplementary to the ordinary numeration system, a symbolical language or notation, which consists of general symbols for numbers, together with symbols of operation and of relation, commonly called signs.

Algebra is thus a symbolical extension of Arithmetic.

Algebra probably originated in India, like the common numeration system, and like it, came to us through the Arabs. It has since received many improvements.

(2) Algebraical Symbols for Numbers.

3. The general symbols employed in Algebra to stand for numbers are letters, in the first instance those of our own alphabet. When we say that a letter, as a, stands for a number, we mean for any number, in the same sense as is conveyed by the words 'a number' in the proposition, 'Twice the double of a number is four times that number,' a proposition which in algebraical language might be written thus, $2 \times 2a = 4a$.

4. As we have learnt in Arithmetic to extend the notion of a number to include fractions, we suppose a to denote any number, integral or fractional. And amongst the particular numbers which it may represent, we include 0 or zero.

5. **Caution.** When any symbol, as a, is used repeatedly in a connected course of reasoning, it means the *same* number throughout.

6. The multiples of a are denoted by $2a$, $3a$, $4a$, &c.

7. If we desire to speak of two or more numbers, we introduce the symbols b, c, &c. to which the same remarks apply.

8. The basis of the rules of combination of these symbols must be sought in Arithmetic, of which Algebra is only a generalization.

9. The rules of combination of symbols that are perfectly general will, when settled, apply to all particular values, whether (i) *known*, or (ii) *unknown*, since whatever is general includes all particular cases.

(3) Applications of Algebra.

10. Algebra may be said to have three principal applications:

i. To obtain general results. Any Rule in Arithmetic comprises an unlimited number of questions which are all one in principle, though differing in details. Arithmetic must deal with particular numbers, and so cannot exhibit a solution in a general form, but Algebra obtains a general result in terms of general symbols, as a, b, c, &c., representing known numbers, and this result applies directly to all particular cases, when the appropriate values are assigned to a, b, c.

ii. To discover *unknown* numbers, i.e. numbers the values of

which are indirectly fixed by given conditions, but which are not otherwise known. It is customary and convenient to denote such unknown numbers by the final letters of the alphabet, x, y, z.

iii. To deduce from the forms of results, expressed in general symbols, the theory of processes, and especially of inverse processes, and to discover new methods of procedure.

This work will be so arranged as to illustrate successively each of these departments of Algebra.

11. In Algebra, numbers and magnitudes are spoken of as 'quantities' and their symbols as 'symbols of quantity;' but all quantity must be thought of as being expressed numerically in terms of some unit, abstract or concrete. And it is in general best to regard all the symbols as representing pure numbers, mentally detaching from them at the outset, when they are concrete, their concrete unit, and reserving it to be brought in again at the end, or cited in the result as the unit of account.

12. All the ordinary cautions learnt in Arithmetic are to be understood to apply in Algebra (until the contrary is stated), as, for example, that Addition and Subtraction only have place with things of the same nature.

(4) Signs of Operation.

13. The fundamental signs of operation are $+$ (plus), $-$ (minus), $\times$ (the multiple cross), and $\div$, which are respectively the signs of addition, subtraction, multiplication, and division.

14. These signs are now commonly used in Arithmetic, on account of their convenience in indicating operations, but they do not belong to the actual processes of Arithmetic and might be dispensed with. They are used in Algebra with the same fundamental meanings as in Arithmetic, but they receive a wider interpretation in Algebra than in Arithmetic. And they are essential to the operations of Algebra.

15. Besides the above signs there are some others, also commonly used in Arithmetic, viz. those of relation, $=$ (equals, or is equal to), $>$ (is greater than), $<$ (is less than), the signs of ratio and proportion : :: :, the bracket ( ) used to signify that whatever

is thus inclosed is regarded collectively, as one quantity, and lastly the convenient abbreviations ∵ (because), ∴ (therefore). Some others will be mentioned later.

(5) The sign +.

16. In Arithmetic the sign + between two numbers, as $3+2$, or 3 yds. + 2 yds., is equivalent to bidding us add them together, and accordingly we do add them together, and say that their sum is 5 or 5 yds. So also in Algebra, if the sign + connects two (or more) multiples of the same quantity, as $3a+2a$, we add them together, precisely as if they were referred to a common unit in Arithmetic. For three a's and two a's make five a's, which we write $3a+2a=5a$. So again $3a+2a+a=6a$.

Quantities which are to be added together are called addenda.

17. Quantities that have a common symbol, and only differ in the numerals, as $3a$, $2a$, a, are called *like* quantities. The numerals are called the numerical coefficients, or briefly, the coefficients of a.

18. Numerical coefficients may be integral or fractional.

19. When a is taken once only we write it simply a, without showing any coefficient, but in adding, &c. we have to remember that its coefficient counts one, as if it were $1a$.

20. The general rule for the addition of like quantities is: Write down the common symbol and prefix the sum of the coefficients.

Exercises. No. 1.

Example of Addition. $4a+3a+a=8a$, by adding the coefficients.

Add the following:—

(1) $3a+5a+7a$.
(2) $15b+16b+b$.
(3) $7b+9b+11b$.
(4) $17c+21c+81c$.
(5) $\frac{1}{2}d+\frac{3}{4}d$.
(6) $\frac{1}{3}e+\frac{1}{5}e$.
(7) $\frac{1}{7}f+\frac{1}{3}f+\frac{1}{21}f$.

21. To show how such forms arise let it be required to think of a number, and to add to it twice the same, and again five times the same and to express the result algebraically.

If the number be supposed to be known we should call it a,

then we have $a+2a+5a=8a$; if unknown, we call it x, and we have $x+2x+5x=8x$. a may be any number, but we are supposed to know what it is; x means a determinate number, but we do not know what.

Exercises. No. 2.

Express algebraically, as in the above examples, and find the sum of—

(1) Twice a number (known) and half the same and two-thirds of the same.

(2) Three times a number (unknown) and four times the same and one-fourth of the same.

(3) If I can walk a miles an hour, how far can I walk in 3 hours? in 5 hours? in $2\frac{1}{2}$ hours? and in the sum of these times?

(6) Use of + between Unlike Quantities.

22. We come now to another kind of Addition, which is properly Algebraical addition, as distinguished from Arithmetical. Suppose we have to add together $3a$ and $5b$, that is two unlike quantities, we write down the work we are directed to perform thus, $3a+5b$, and having done that we can do no more. For a and b being different symbols, we cannot unite these quantities into one.

23. Considered arithmetically, this is an operation indicated, but not performed, but in Algebra we give such an extended meaning to the symbols that we take an operation indicated as being actually performed. So this is considered to be Addition.

24. Thus we see that, +, which was in Arithmetic only a convenient substitute for words, is in Algebra a necessity, for it often constitutes the operation itself.

25. An algebraical quantity and a particular number, represented by a figure, are unlike quantities. $3a$ and 7 are added as $3a+7$.

26. We have learnt in Arithmetic that the order of addenda is indifferent: thus

$a+b=b+a$, and $a+b+c=b+c+a=c+a+b=$ &c.

But, although the order of arrangement does not affect the total, symbols are usually written, when there is no reason to the contrary, in the order of the alphabet.

27. The complete rule of Addition of single quantities will now be: Add together all like quantities and write down one after another their sums, and also all the unlike quantities, connecting them by the sign +.

Exercises. No. 3.

Examples of Addition.—

The sum of $4a$, $5b$, and $2c = 4a+5b+2c$.

The sum of $3a$, $2b$, $2a$, $3b$, and $c = 5a+5b+c$.

Add together:—

(1) $2a$, $3b$, and c.

(2) $3a$, $4c$, and $6d$.

(3) $4c$ and 2 and $3e$.

(4) $5d$ and 7.

(5) $3a$, $4c$, $2a$, d, and 3.

(6) $2a$, $3b$, $2b$, 3, $5a$, and 4.

Express algebraically:—

(7) The sum of two known numbers.

(8) The sum of two unknown numbers.

(9) A known number increased by 7.

(10) An unknown number increased by 5.

(11) A known number and twice the same together with 15.

(12) Twice an unknown number together with one-third of the same number, to which add 10.

(13) a being a boy's age, express the age of his mother who was 25 when he was born, and that of his father who is half the boy's age older.

28. Since a, b, c are known numbers, we may if we please give them particular values, but we cannot give values to unknown quantities, since their values are determinate, though at first unknown to us. We should be nearly sure to give them the wrong values.

Exercises. No. 4.

Example.—

If $a = 2$, $b = 3$, $c = 5$, find the value of $5a+11b+2c$.

Required value $= 5\times 2+11\times 3+2\times 5 = 10+33+10 = 53$.

Required the values of (1) to (6) in Exercises, No. 3, if $a = 3$, $b = 2$, $c = 1$, $d = 0$.

(7) The sign —.

29. In Arithmetic the sign — between two numbers is a direction that we are to take the latter (called the *subtrahend*) from the former (the *minuend*), and accordingly we do so. Thus $5-2=3$, and £5−£2 = £3. So, in Algebra $5a-2a=3a$, exactly as if the quantities $2a$, $5a$ were referred to a common denomination or unit in Arithmetic. But if we are required to subtract $3b$ from $5a$, we can only write $5a-3b$, and this is accepted as subtraction. So also if we are to subtract 7 from $3a$, the result, or remainder, is $3a-7$.

30. We shall see as we proceed that the remarks made in (23) respecting the extended meaning given in Algebra to the sign + apply with still greater force to the sign —.

Exercises. No. 5.

Example.—From $5b$ take 9. Remainder $5b-9$.

(1) From $3a$ take $2b$.

(2) From $5b$ take 17.

(3) From the sum of $5a$ and $9a$ take $11b$.

(4) Subtract c from 35.

(5) From the sum of 6 and 15 subtract $2x$.

(6) What is the excess of $3d$ over $2c$?

Express algebraically:—

(7) Twice a known number less 3.

(8) Three times a known number less half another known number.

(9) Five times an unknown number diminished by one-third of the same.

(10) Four times an unknown number diminished by 15.

(11) Eight times an unknown number diminished by 3 times a known number.

(12) A known number diminished by five times another (unknown) number.

(8) The Sign ×.

31. In Arithmetic the sign × (the multiple cross, which we read 'into') when placed between two or more numbers, then called factors, as $2 \times 3 \times 4$, indicates that these factors are to be multiplied together, and accordingly we do multiply them together, and say that their product is 24. In Algebra $a \times b \times c$ stands for the product of the factors a, b, c, but × is omitted whenever this can be done without ambiguity. Thus we do not write $3 \times a$, but $3a$ (read 'three a'), and $a \times b \times c$ is usually written abc, and so read.

32. But the multiple cross cannot be omitted from $2 \times 3 \times 4$, since it would then become 234, and would mean two hundred and thirty-four; it is however, here and elsewhere, often replaced by a single point, which may be called the multiple point, placed on the line, thus, 2.3.4 (read as before, 'two into three into four'). As the decimal point is sometimes placed in this position, care must be taken not to use the multiple point in any case where it might be mistaken for the decimal point, not therefore between two figures, as 2·3, which might be mistaken for 2·3.

33. We have learnt in Arithmetic that the order in which the factors of a product are taken does not affect the value of the product. Hence $ab = ba$, and $abc = bca = cab =$ &c.

34. If any of the factors are common numbers, we may change the order of the factors so as to bring them together and effect their multiplication. Thus $2a \times 3b = 2 \times 3ab = 6ab$.

Exercises. No. 6.

Example.—

Multiply $3a$ by $5b$. Answer $3a \times 5b = 3 \times 5ab = 15ab$.

(1) Multiply $2b$ by $3c$.

(2) Simplify $1.2.3.4 \times 6a \times 7b \times 8c$.

(3) Simplify $2 \times 6a \times 3b + 3 \times 4b \times 2a + 14ab$.

Express algebraically:—

(4) Three times a known number multiplied by twice a known number and diminished by 6.

(5) Five times a known number multiplied by an unknown number, the whole increased by 11.

(6) Show by a figure that $ab = ba$, when a, b are particular integers, and generalize for any integers.

(9) The Sign ·

35. The sign $\div$ between two numbers indicates in Arithmetic that the former (called the *dividend*) is to be divided by the latter (the *divisor*). The result is called the quotient. Thus $24 \div 6 = 4$. We have learnt also to attach a meaning to the division $2 \div 3$ (see *Scholar's Arithmetic*, V. 1), viz. the fraction $\frac{2}{3}$. In Algebra also $a \div b$ indicates that a is to be divided by b (and is read 'a divided by b'), but the operation $a \div b$ is most commonly represented by the fraction $\frac{a}{b}$ (read 'a by b'). Numerical fractions should be reduced to lowest terms, as in Arithmetic.

36. Observe that $\frac{2a}{7}$ is the same as $\frac{2}{7}a$, by (33), for we may treat $\frac{1}{7}$ as a factor, so that $\frac{2a}{7} = 2 \times \frac{1}{7} \times a = \frac{2}{7}a$.

Again, any factors which are the reciprocals of quantities, as $\frac{1}{b}, \frac{1}{c}$, may change their order amongst themselves and with others without affecting the result. Thus $\frac{adef}{bc} = \frac{aefd}{cb}$ = &c.

Exercises. No. 7.

Example.—Divide $12b$ by $15c$. Quotient $= \frac{12b}{15c} = \frac{4b}{5c}$.

(1) Divide $15a$ by $3b$.

(2) Multiply $2a$ by $4b$ and divide by $16c$.

(3) Simplify $3a \times 4b \times 6c \div 2a \times 3b \times 5d$.

Express algebraically:—

(4) The quotient of a known number divided by 6.

(5) The quotient of an unknown number divided by three times a known number.

(6) An unknown number multiplied by a known number and divided by another unknown number.

(10) Powers and Indices.

37. The forms of products are further simplified as follows, when two or more factors are alike. The product $a \times a$ is represented by a^2, $a \times a \times a$ by a^3, $a \times a \times a \times a$ by a^4, &c., the small figure, called the *index* (or the *exponent*), showing the number of factors which are alike. a^2 is called the second power of a, or its square (and is read 'a square'); a^3 is called the third power of a, or its cube (read 'a cube'); a^4 the fourth power of a (read 'a to the fourth'), &c.; a^n the n^{th} power of a (read 'a to the n^{th}'), which means that the factors $a \times a \times a \times \ldots \times a$ are n in number. a itself is reckoned the first power of a or a^1, and the index 1 is understood whenever the index of a has to be combined with other indices.

Hence $a \times a \times a \times b \times b = a^3 b^2$, and $\dfrac{a \times a \times a \times a \times c}{b \times b \times d} = \dfrac{a^4 c}{b^2 d}$.

38. The following are the rules for indices. The continued product of m a's multiplied by the continued product of n a's makes the continued product of $(m+n)$ a's, which is denoted by a^{m+n}. Hence $a^m \times a^n = a^{m+n}$. Again, the continued product of m a's divided by the continued product of n a's $(m>n)$ is a fraction with m a's in the numerator and n in the denominator, and the quotient is the excess of the number of a's in the numerator above those in the denominator, which is $m-n$.

$$\therefore \quad \frac{a^m}{a^n} = a^{m-n} \; (m>n).$$

39. Again, $(a^2)^3$ means that a^2 or $a \times a$ is to be repeated three times, or $\overline{a \times a} \times \overline{a \times a} \times \overline{a \times a}$. Thus, the factor a is repeated 2×3 times, or 6 times, and the product $= a^6$; so that $(a^2)^3 = a^6$, and, generally, $(a^m)^n$ is m a's repeated, as factors, n times, or mn factors, making a^{mn}; $\quad \therefore \; (a^m)^n = a^{mn}$.

Hence $(a^3)^2 = a^6 = (a^2)^3$, and $(a^n)^m = a^{mn} = (a^m)^n$.

40. **Caution.**—The beginner often confounds $(a^3)^2$, which is a^6, with $a^3 \times a^2$, which is a^5; or $(a^2)^5$, which is a^{10}, with $a^2 \times a^5$, which is a^7. Notice carefully the difference. $a^3 \times a^2$ is one factor repeated three times and then two more added; $(a^3)^2$ is a set of

three factors repeated twice: in the latter case there is multiplication of the number of factors, in the former only addition.

Exercises. No. 8.

Example.—Simplify $a^2 \times b^3 \times a \times 2b$. Collecting the indices of like factors, we have, Answer, $2a^{2+1}b^{3+1} = 2a^3b^4$.

Find the products of the following :—

(1) $a \times a \times b$.
(2) $a \times b \times a \times b$.
(3) $a \times a \times c \times b \times a \times b$.
(4) $(a^5)^3$.
(5) $(b^3)^2$.
(6) $(b^4)^2 \times (c^2)^2$.
(7) $(b^2)^3 \times (b^3)^4$.
(8) $(a^2)^6 \times (a^3)^{12} \times (b^3)^2$.
(9) $b \times c^2 \times (c^3)^2$.
(10) $c^2 \times (c^3)^2 \div (c^2)^4$.
(11) $a^m \times a^n \times a^p$.
(12) $a^a \times a^b$.
(13) $a^p \times a^{2q} \div a^q$.

(11) Roots and Radicals.

41. The square root of a number is that number which multiplied into itself will produce the first-mentioned or primary number.

The square root of a is denoted by $\sqrt{a}$, sometimes called a *radical*. $\surd$ (called 'the radical sign') was originally the initial letter of the Latin word *radix*, 'a root.'

By definition $\sqrt{a} \times \sqrt{a} = a$. Thus, since $2 \times 2 = 4$, 4 is the square of 2, and inversely, 2 the square root of 4.

42. The cube root of a, denoted by $\sqrt[3]{a}$, is such that

$$\sqrt[3]{a} \times \sqrt[3]{a} \times \sqrt[3]{a} = a.$$

Thus, since $2 \times 2 \times 2 = 8$, 8 is the cube of 2, and, inversely, 2 is the cube root of 8. So with the fourth root, &c. The general name is the n^{th} root, denoted by $\sqrt[n]{a}$, such that

$$\sqrt[n]{a} \times \sqrt[n]{a} \times \ldots \sqrt[n]{a} \ (n \text{ factors}) = a.$$

(12) The Signs of Affection. Positive and Negative.

43. It may be remarked that such forms as $5ab$, $3ab^2c$, $\frac{2a^2b}{c^3}$, &c., being combinations so much closer than $5 \times a \times b$,

$3 \times a \times b \times b \times c$, and $2 \times a \times a \times b \div c \times c \times c$, which they represent, the former have practically superseded the latter as Algebraic quantities, thus almost thrusting back the signs $\times$ and $\div$ into the status which they held in Arithmetic of mere indicators of operations to be afterwards carried out.

44. Partly perhaps on this account + and − stand out pre-eminently as 'the signs' in Algebra, and the name is often applied to them as exclusive of all others. They are further described as 'signs of affection,' as if they bestowed a certain quality, which is what is here meant by the word 'affection,' upon the quantities which they precede. + is called the *positive* sign, and − the *negative* sign, and all quantities are divided into *positive quantities* and *negative quantities*, according as they are affected, as it is expressed, with the sign + or the sign −.

45. In this division quantities which appear without a sign, as a, b, are positive, and the sign + may be supplied to them whenever their sign comes in question. + is not generally written before the first of two or more quantities, as we see in $a+b=b+a$, but here it is taken for granted by the fact that each quantity has + in its turn, that we might write $+a+b = +b+a$. In fact, a may be considered to be the same as $0+a$, as if we started from 0 and obtained a by adding it. A negative quantity, such as $-2a$, is said to have the negative coefficient -2, the sign being attached to the coefficient, not to the symbol.

CHAPTER II.

ALGEBRAIC EXPRESSIONS.

(1) Simple and Compound Expressions.

46. The general name for a collection of symbols, connected by signs of affection, as $3ab+2bc-5cd$, is 'an expression,' and the separate parts, $3ab$, $+2bc$, $-5cd$ (beyond which the effect of each sign does not extend), are called 'terms.' Thus an

expression answers to a phrase, and a term to a word, in ordinary language.

47. An expression is either *simple*, that is, consisting of one term only, as ax, or *compound*, consisting of two or more terms. Compound expressions, when they consist of two terms, are called binomials, as $ax+by$; if of three, trinomials, as $ax+by+cz$; if of more, multinomials; but these names will not be of frequent occurrence at present.

When we speak of an expression, we are thinking more especially of the symbols before us: we call it a quantity, simple or compound, when we identify it with that which it represents.

(2) Terms and Dimensions.

48. A term, or single quantity, is either *simple*, as $3a$, having only one symbolical factor (we do not count the numerical factor), or composite, as $3abc$ and $4ab^2c$, which have several.

49. Each factor is reckoned a dimension, the name being taken by analogy from the properties of space, quantities of one dimension corresponding to linear magnitudes, those of two dimensions to the magnitudes of superficial or square measure, and those of three dimensions to the magnitudes of solid or cubic measure. It is proved in Arithmetic (See *Scholar's Arithmetic*, IX. 8, 10) that the number of linear, superficial, or solid units in magnitudes of these respective kinds depends respectively on one, two, or three factors representing length only, or length and breadth, or length and breadth and height. A larger number of factors can have no corresponding analogy in the properties of space, but the name dimension is extended from the first three to any number of factors[1].

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[1] It remains true in Algebra as in Arithmetic that in the proper sense of Multiplication (Numerical Multiplication) there can be no such thing as a concrete multiplier. We cannot multiply by 3 ounces, or by 3 pence, or by 3 pints and in a composite quantity, as $3abc$, we can only attribute a concrete unit to some one factor (the multiplicand), *and the unit, whether abstract or concrete, must remain the same in the product as in the multiplicand.*

But in dealing with the properties of space we have the following rules for estimating extension:—

i. *Superficial extension.* The number of square inches (or other square units)

50. Two terms are said to be 'homogeneous' when they are of the same dimensions, as $3a^2b$ and $4abc$, and an expression is said to be homogeneous when *all* its terms are of the same dimensions, as $3a^2b+4abc+2b^3$. This is a point which we learn habitually to notice, as an expression originally homogeneous will, unless multiplied by a factor which is not so, remain homogeneous throughout an investigation, and any failure of homogeneity is a token of error. For it is like adding a superficial foot to a solid cube. In reckoning dimensions the numerical coefficients do not count.

(3) Mode of reading Algebraic Expressions.

51. Some practice in reading Algebraic expressions and in writing them from dictation will now be found useful. A few preliminary examples follow.

We read $5a-16c+15d$, five a minus sixteen c plus fifteen d; $3a^2-4ab+6b^2$, three a square minus four ab plus six b square; $7a^2b-9ab^2+14b^3$, seven a square b minus nine ab square plus

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in a rectangle is equal to the product of the numbers expressing the inches (or other *linear* units) in the length and breadth.

ii. *Solid extension.* The number of cubic inches (or other cubic units) in a right solid is equal to the continued product of the numbers expressing the inches (or other linear units) in the length, breadth, and height.

This is clearly different from numerical multiplication, in particular, by its conducting to units of a different nature from the primary unit, linear units giving rise to square or superficial units where there are two factors, and cubic or solid units where there are three factors. Still, it so naturally suggests multiplication, and indeed has so close a connection with multiplication of fractions, that, so soon as these differences are distinctly perceived, it might fairly be regarded as a species of Geometrical Multiplication, as it practically is whenever we speak of length × breadth, length × breadth × height.

Hence the idea of dimensions.

We see that ab, in addition to its primary meaning of the product of two numbers, admits of a secondary interpretation: ab may mean (in the sense just explained) the size of the rectangle of which the sides are a and b; a^2 the square of which the side is a; abc the right solid of which the sides are a, b, c; a^3 the cube of which the side is a, all expressed in their proper dimensional units.

Supposing the kind of extension used in the Second Book of Euclid in forming the rectangle AB, CD to be called (under proper explanations) Geometrical Multiplication, the above might be called Dimensional Multiplication.

fourteen b cube; $6a^3bc - 13ab^2c^2 + 5d^5$, six a cube bc minus thirteen a b square c square plus five d to the fifth.

Exercises. No. 9.

Read the following:—

(1) $7a^2 - 5b^2 + 3c^2$.
(2) $3ab^3 - 2abc^2 + 4bc^3$.
(3) $7bc^4 - 5c^3d^2 + d^5$.
(4) $8x^5 - 3x^3y^2 + 2xy^4 - y^5$.
(5) $3ax^5 - 4by^5 + cx^2y^3$.

Exercises. No. 10.

Write down the following:—

(1) Five a plus nine c minus three d.

(2) Fifteen c minus seventeen d plus eighteen e.

(3) Eighty-one b minus thirteen e plus fourteen f minus nineteen g.

(4) Seventeen x plus twenty y minus thirteen z.

(5) Three a square plus four ab plus six b square.

(6) Seven a square minus six ab plus nine b square.

(7) Five a square b minus three a b square plus six b cube.

(8) Five a to the fourth plus four a cube minus three a square b plus seven a b cube plus nine b to the fourth.

Find the value of the above two sets of expressions, when the following particular values are assigned, viz. $a = 3$, $b = 2$, $c = 4$, $d = 3$, $e = 5$, $f = 6$, $g = 10$, $x = 2$, $y = 1$, $z = 11$.

CHAPTER III.

COMBINED USE OF THE POSITIVE AND NEGATIVE SIGNS.

(1) Extension of the Meaning of Addition.

52. When an expression contains several like quantities, or, as we shall now call them, like terms, they must be put together to form one term. If the like terms are all positive, as $2b + 3b + 5b$,

we have seen that they add together and make $10b$; if they are all negative, as $-2b-3b-5b$, we think of them as quantities to be taken away in succession from some other quantity, either in the question before us, or when the occasion arises. To take them away successively is the same in effect as to take away by one subtraction their aggregate, or the sum of the terms considered absolutely (i.e. irrespective of sign), which is $10b$. When we prefix the sign of subtraction, we have the result $-10b$. If some terms are positive and some negative; as in (i) $15b+3b-2b+6b-8b-5b$, and (ii) $15b-12b+3b+6b-8b-9b$, the beginner may set the positive and negative quantities under each other in two columns, marking over them + and −; add them up separately, and finally subtract, if possible, the sum of the second column from that of the first, as in (i), where we have the result $9b$. Suppose, however, the second column to have the larger total, as in (ii), it is no longer arithmetically possible to subtract it: we have in fact $24b-29b$. But Algebra gives this general rule. When two like quantities have different signs, take away that which is absolutely (i.e. irrespective of sign) the less from that which is absolutely the greater, and prefix to the difference the sign of the greater. Here then we have $-5b$. This shows that there is a predominance on the whole, to the extent of $5b$, of the quantities to be taken away.

(i)		(ii)	
+	−	+	−
$15b$	$2b$	$15b$	$12b$
$3b$	$8b$	$3b$	$8b$
$6b$	$5b$	$6b$	$9b$
$24b$	$15b$	$24b$	$29b$
$15b$			$24b$
$9b$			$-5b$

53. This separation of the positive and negative quantities may be compared to keeping a Dr. and Cr. account of receipts and expenses and ascertaining the balance. And this may, of course, fall on either side. And even when we do not collect the positive and negative quantities into separate columns, this is the way in which we think of them. And we thus see how to answer the question, Are negative quantities real quantities? They are real, as matters of account; not as having any real separate existence.

54. The arrangement in separate columns is only used at first. We have to learn to begin at one end (choosing the most convenient) when the quantities are arranged in line, as

$$15b+3b-2b+6b-8b-5b,$$

and collect the coefficients one by one, adding them in when + and deducting them when −. In this case we begin at the beginning (or top of the column), since the positive quantities seem to predominate at that end, and the steps of addition are as follows: 15, 18, 16, 22, 14, 9*b*. We reserve the symbol to be named, once for all, at the end.

$$\begin{array}{r} 15b \\ +\ 3b \\ -\ 2b \\ +\ 6b \\ -\ 8b \\ -\ 5b \\ \hline 9b \end{array}$$

The process of finding this value is still called adding the terms together, although every negative sign involves an arithmetical subtraction. The result, whether positive or negative, is called the 'algebraical sum' of the terms.

The rule for putting together a number of terms, or, as it is commonly called, 'reducing them to a single term,' may be now stated thus:—*Write down the common symbol and prefix to it the algebraical sum of the terms.* This common symbol may be simple or composite.

Exercises. No. 11.

Add together:—

(1) $13a+5a-4a-17a+21a$.

(2) $-5b+6b-18b$.

(3) $17x-2x+9x-5x-15x+4x$.

(4) $-3y-2y-8y+6y-3y+15y$.

(5) $7az+3az-5az-8az+5az-13az$.

(6) $7a^2b+3a^2b-5a^2b-8a^2b+5a^2b-13a^2b$.

(7) $9a^3xy-5a^3xy+13a^3xy-17a^3xy$.

(2) Interpretation of Negative Quantities.

55. As a further illustration of the employment of positive and negative quantities let the cases (i) and (ii) in (52) be imagined to arise out of some game, where an account is kept by means of two different sets of counters, say white and red, white always answering to positive and signifying gains, and red answering

to negative and signifying fines. Such an account would be closed by adding up gains and losses, either cancelling off by degrees red counters against white, or by at once reckoning all the whites and all the reds and taking the difference. Either way there will remain the same balance, which may be of either colour, and will need some adjustment, according to the rules of the game. Plainly red counters and white are in themselves equally real, and in reckoning them we are equally bound by the laws of counting and of Arithmetic, and the balance has in either case an equally intelligible meaning. This further illustrates what we have said, that negative quantities are real as matters of account.

56. We may easily see also from this illustration the effect of any individual counter. What would it be to add a white counter, of intrinsic value c, to any score? Clearly, to increase it by c. To add a red counter, value c, would be to diminish the score by c, since it implies a fine of that amount. To take away a white c, would diminish the score by c, whilst to take away a red c would have the effect of increasing the score by c, by taking off a penalty of that amount. Whence we see that *algebraically* to add a negative quantity is the same thing as to subtract a positive quantity of the same value, and to subtract a negative quantity is the same as to add a positive quantity. These results are comprised under the Rule:—To add an algebraical quantity, take it with its sign of affection; to subtract an algebraical quantity, change its sign (i.e. + into −, or − into +) and add it.

57. The use of such quantities as $2a-3b$, where a and b are perfectly general, compels us to recognize negative quantities in calculation. Their general signification, when they appear as a result, is that we have stated a case in which those quantities which on the face of the enunciation were supposed to be the less in amount have turned out to be the greater. Thus, if asked for the difference of $2a$ and $3b$, we should certainly put $3b$ first if we knew it to be the larger; but if we put $2a$ first, under a false supposition, $2a-3b$ will certainly come out negative for certain values of a and b. This not only shows us that $3b$ is the larger, but it shows us exactly how much larger it is. And this

serves our purpose just as well as if we had arranged the numbers in the inverse order. In money accounts, as a positive quantity signifies an addition to receipts, and a negative quantity a diminution of the amount in hand, so a positive balance means money in hand, and a negative balance money due.

58. In Geometry, we may give a meaning to negative quantities of a still more real and tangible kind. If we make measurements along a line from a given point, as, Ten paces forward, two back, three more back, four more back, and five more back: the application of these negative quantities would finally land us in a negative result; viz. on the whole, four paces backwards from the starting-point. And we find that we can always give this consistent interpretation to quantities affected with the two signs. A positive quantity denotes a measurement of so many units along a line, measured from a given point in a given direction: a negative quantity denotes a measurement of so many units along the line produced in a direction opposite to the former.

(3) Brackets.

59. When an algebraic expression is to be treated as a whole, we place it within brackets, which are of various forms, as $2(b+c)$, or $-3\{b-c\}$, or $4c[d-e]$, or we draw over it a line, called a 'vinculum,' as $3b.\overline{b-c}$. In all these cases any sign or factor outside the bracket applies to the whole.

For example, if $b=5$, and $c=3$, $2(b+c)=2(5+3)=2\times 8=16$, and $-3\{b-c\}=-3\times 2=-6$.

60. Let it be required to examine the meaning of the following expressions:—

i. $a+(b+c)$. ii. $a+(b-c)$. iii. $a-(b+c)$. iv. $a-(b-c)$.

i. $a+(b+c)$ bids us add to a the sum of b and c: we obtain the same result by adding the whole or the parts separately,

$$\therefore\ a+(b+c)=a+b+c.$$

For example, $7+(5+3)=7+8=15=7+5+3$.

ii. $a+(b-c)$ bids us add to a the excess of b over c. If we add b, we add too much by c, and must again deduct c,

$$\therefore\ a+(b-c)=a+b-c.$$

For example, $7+(5-3)=7+2=9=7+5-3$.

iii. $a-(b+c)=$ bids us subtract from a the sum of b and c, which is the same as taking them away in succession;

$$\therefore\ a-(b+c)=a-b-c.$$

For example, $9-(5+3)=9-8=1=9-5-3$.

iv. $a-(b-c)$ bids us take from a the excess of b over c. If we take away b, we shall have taken away too much by c, and this must be restored. Hence $a-(b-c)=a-b+c$.

For example, $9-(5-3)=9-2=7=9-5+3$.

61. From the above investigation we derive the following rules for dealing with the signs of brackets. Since $+(b+c)$ gives $+b+c$, and $+(b-c)$ gives $+b-c$; *when the bracket is preceded by* $+$, *the brackets may be removed, and all the signs may be left as they are.* Again, since $-(b+c)=-b-c$, and $-(b-c)=-b+c$; *when a bracket is preceded by* $-$, *the brackets may be removed, provided we at the same time change the signs of all the terms within the brackets.*

Conversely, *Any quantities whatever may be placed within brackets, retaining their own signs, if* $+$ *be placed before the bracket, and any quantities whatever may be placed within brackets, their signs being all changed, if the sign* $-$ *be placed before the bracket.*

62. We may remark that $a+(b-c)$ and $a-(c-b)$ have the same value, for the difference of order $a+b-c$ and $a-c+b$ does not affect the value. In other words supposing we have misconceived the relative magnitude of b and c, so that to add $b-c$ is equivalent to subtracting $c-b$, our misconception makes no difference in the end. This is one example of what gradually becomes clear, that the rules of Algebra are independent of the magnitudes of the respective quantities, for when dealing with general symbols we obtain results, *the form of which is independent of all questions of value.* This principle is part of a wider one, which has been called 'The principle of the permanence of equivalent forms.'

Since the effect of the sign $+$ or $-$ in $a+(b+c)$ or $a-(b+c)$ extends throughout the bracket, $(b+c)$ must be considered a term, by (46). It may be called a compound term.

63. Having recognized the free use of negative quantities, we must now extend the description of general symbols in (4), thus, a may denote any number, integral or fractional, positive or negative.

But observe that we call $-3b$ or $-3(b-c)$ negative terms, on account of their being negative in form, and so apparently negative, without any reference to the intrinsic values of b or $(b-c)$. But if negative values were assigned to b or to $(b-c)$, they would turn out to be for those values intrinsically positive. As we have said before, Algebra looks at the form of the expression, not at intrinsic value.

64. An index placed over a bracket applies to the whole quantity within the brackets.

Thus, $(ab)^2 = ab \times ab = a^2b^2$;

$(abc)^2 = abc \times abc = a^2b^2c^2$;

$(a^2bc)^3 = a^2bc \times a^2bc \times a^2bc = a^6b^3c^3$.

And we get this rule:—We may remove the brackets from a *simple expression*, which is composite, if we apply the index to every factor separately. Thus, $(abcd)^n = a^nb^nc^nd^n$ and

$$(3a^3b^2c)^3 = 27a^9b^6c^3, \quad (3a^3b^2c)^n = 3^na^{3n}b^{2n}c^n.$$

65. **Caution.**—But *this cannot be done with a compound expression*, as $(a+b)^2$ or $(a+b)^3$, which we shall see hereafter has to be treated quite differently.

66. The sign $\sqrt{}$ has the effect of a bracket, the line above being in fact a vinculum: $\sqrt{a+b} = \sqrt{(a+b)}$, and is the root of the sum of $(a+b)$, which is different from $\sqrt{a} + \sqrt{b}$. Thus, $\sqrt{9+16} = \sqrt{25} = 5$. This is quite different from

$$\sqrt{9} + \sqrt{16} = 3+4 = 7.$$

67. Another arrangement, which has all the effect of a bracket and which is governed by the same laws, is the fractional; the bar which separates the numerator and denominator binding together the whole expression in the numerator like a vinculum.

Thus, $a - \frac{b-c}{3} = a - \frac{b}{3} + \frac{c}{3}$, just as if it were written $a - \frac{1}{3}\overline{b-c}$.

So again, $a-\frac{2b+3c-4d}{c}=a-\frac{2b}{c}-3+\frac{4d}{c}$.

Caution.—The most careful attention should be given to these rules about brackets: carelessness in regard to them is the most frequent cause of mistakes with beginners in Algebra.

Examples of bracketing:—

(1) To remove the brackets from $(a-3b+2c)+(b-6c)$. The signs before the brackets being +, we have nothing to do but to take away the brackets, and we have

$$a-3b+2c+b-6c=a-2b-4c, \text{ when reduced.}$$

(2) To remove the brackets from

$$(3a-5b+7c)-(6a-7b+8c-9d).$$

We change the signs of all the terms in the second bracket, and we have $3a-5b+7c-6a+7b-8c+9d=-3a+2b-c+9d$, when reduced.

(3) To put $-3a+2b-c+9d$ into brackets, beginning successively with the first, second, and third terms, and with the signs of these terms outside. We have the following results:—

$$\begin{aligned}-3a+2b-c+9d&=-(3a-2b+c-9d);\\&=-3a+(2b-c+9d);\\&=-3a+2b-(c-9d).\end{aligned}$$

Exercises. No. 12.

Remove the brackets from—

(1) $(2ax-3by)-(5ax-by)$.

(2) $6a^2x-(13by^2-5abxy-9y^4)+(7x^3-8y^3)$.

(3) $2-(5-8y+13z)-(3+16y-15x)$.

(4) $2-\frac{15-8y+21z}{3}-\frac{30+80y-15x}{5}$.

(5) $\frac{3ax-5y}{2}-\frac{2ay-\frac{5}{2}x}{a}+\frac{7z-3y}{b}$.

(6) Divide $3a-5b-16c+8d-3f+4h$ in three ways between two vincula, the second beginning successively at the 3rd, 4th, and 5th terms.

(4) Complex Bracketing.

68. Sometimes there are brackets within brackets, like a nest of boxes, one within the other, as

$$10x-\{5a-[4b+2c-(6b-\overline{5c-2d})]\}.$$

In removing these, the beginner should proceed step by step, removing first the innermost bracket, then the innermost bracket next remaining, and so on. The reason for this order is that the beginner should get it firmly fixed in his mind that in a series of successive symbols of operation the left-hand symbol shows the last operation performed, and this holds all through the series. There are some operations in which the order does not matter, but there are others in which it does, and it is well to follow the order which is universally applicable. Thus, $\sqrt{a+(b+c)^2}$ requires that we take the square root of a increased by the square of $(b+c)$, and the operation of taking the square root must come last. It is denoted by the symbol on the left-hand. The above example would be treated thus:

$$\begin{aligned} &10x-\{5a-[4b+2c-(6b-5c+2d)]\},\\ &=10x-\{5a-[4b+2c-\ 6b+5c-2d]\},\\ &=10x-\{5a-\ 4b-2c+\ 6b-5c+2d\},\\ &=10x-\ 5a+\ 4b+2c-\ 6b+5c-2d. \end{aligned}$$

Exercises. No. 13.

Remove the brackets from:—

(1) $2b\{3x-2a+6-[12a-(b+9c-\overline{2x-7a-3})]\}$.

(2) $-[3-4\{6-(-8a+5b-3c)-(7a-\overline{3b-2c})\}]$.

(3) $-3[(a+b)-\{(2a-3b)-(5a+\overline{7b-16c})-(-13a+2b-\overline{3c-5d})\}]$.

(4) $2[-\frac{a-b}{3}-\{\frac{3a-4b}{2}-(6a-\frac{9b-5c}{7})\}]$.

69. It will be found that we may get the correct result by writing down the symbols at first without signs, and then counting the number of minus signs preceding each symbol: where there are no minus signs or an even number, we write +, and for an odd number, −.

(5) The Sign =.

70. The sign = signifies that the expressions between which it is placed are equal: and this equality implies two things, (i) Numerical equality, apart from sign; (ii) that they are affected by the same sign. Thus, $a-b=3$ would not be true if a and b were of the respective values 4 and 7; for in that case we must write $a-b=-3$.

71. Two algebraic expressions connected by the sign = are called 'an equality,' and also (in senses hereafter to be explained) an equation.

When we connect two expressions by the sign = we are said to equate them.

CHAPTER IV.

(1) Addition, &c. of Algebraic Expressions.

72. By the rule of brackets (61)

$(2a+3b-c+d)+(3a-2b+2c-e)$ is equivalent to

$2a+3b-c+d+3a-2b+2c-e$, or $5a+b+c+d-e$,

whence the rule,—To add two or more algebraic expressions, write all the terms after one another with their proper signs and proceed to simplify by collecting all like terms.

73. It comes to the same thing, if instead of writing the expressions in order after one another, we arrange them below one another, placing like terms as much as possible under like, as in the annexed example, and then add up.

$$\begin{array}{r} 2a^2+3b^2-5cd \\ -4a^2-2b^2+6ce \\ 6a^2+9b^2-5de \\ 7a^2-3b^2+5de \\ 8a^2+b^2+ce \\ \hline 19a^2+8b^2-5cd+7ce \end{array}$$

The following are additional examples :—

$$\begin{array}{r} x^2-2y^2 \\ 3x^2+5y^2 \\ 4x^2-6y^2+z^2 \\ 8y^2-2z^2 \\ -7x^2+5y^2-6z^2 \\ \hline x^2+10y^2-7z^2 \end{array} \qquad \begin{array}{r} 4abc-2yz^2+3y^3 \\ 5abc+6yz^2-2y^3 \\ -4abc+yz^2-9y^3 \\ -7abc-3yz^2+y^3 \\ 6abc+yz^2-9y^3 \\ \hline 4abc+3yz^2-16y^3 \end{array}$$

Exercises. No. 14.

Add together the following :—

(1) $a-3b+c$, $2a-3b-5c$, $-3a+5b-6c$, $7a-6b-9c$.

(2) $2x-5y+6$, $2x-3y-9$, $-3x+8y-7$, $5x-3y+2$.

(3) $7xy+8yz-5$, $2xy-3yz+9$, $-7xy+4$, $8yz-17$.

(4) $3-x+y$, $2x-5+9y$, $3y-7+5x$, $2-9y$, $6x+5$.

(5) $7ab^2-cd$, $8cd-5b^2a$, $7ab^2+16cd$, $-8ab^2+15cd$.

(6) $2yz^2-3zx^2+y^3$, $7yz^2+13zx^2+y^3$, $2yz^2+zx^2+19y^3$, $7yz^2+31zx^2-27y^3$.

(7) $30ab+15bc$, $3bc+2ca$, $16ca+5ab$, $7ab-21bc$, $19bc-3ca$, $7ac-15bc$, $16ac-15ab$.

74. **Literal Coefficients.**—If we have an expression of the form $2ax^3-3bx^2+4cx-d$, we speak of $2a$, $-3b$, $4c$ as the coefficients of x^3, x^2, and x. They are said to be literal coefficients, as containing letters, and in that respect differing from numerical coefficients. It is often stated that any term, as $3abc$, may be broken up in any manner into factors, as $3a.bc$ or $3ab.c$, &c., and that the $3a$ is the coefficient of bc, $3ab$ of c; but it should be understood that this is not done in an arbitrary manner, without a reason. Practically it is only when some quantity stands out as that with reference to which the term is classified that we speak of its multiplier, the factor which completes the term, as its coefficient. For instance, it is usual to arrange expressions which contain different powers of the same letter, so that the powers of that letter either increase term by term or else diminish. Thus,

$$3a^3+2a^2b+5ab^2+7b^3,$$

is said to be arranged by descending powers of a, in which case 3,

$2b$, $5b^2$, $7b^3$ are the coefficients of the successive powers. But the same expression may be regarded with respect to b, in which case it is said to be arranged by ascending powers of b, and the coefficients of the successive powers are $3a^3$, $2a^2$, $5a$, and 7. The $7b^3$ in the first instance and $3a^3$ in the second are said to be independent coefficients, because they do not contain the letter of reference. But this way of speaking applies particularly to the mixture of known and unknown quantities, as in

$$2ax^3-3bx^2+4cx-d.$$

One expression is said to be a higher expression than another, when it contains a higher power of the letter of reference.

In adding several expressions of this kind, we collect the coefficients of like powers in a bracket.

Thus, $(2ax^3-3bx^2+4cx-d)+(-ax^3+2bx^2+5cx+d)$
$+(5bx^3+6cx^2+dx+e)+(7cx^3-9)=$
$(a+5b+7c)x^3+(6c-b)x^2+(9c+d)x+e-9.$

Exercises. No. 15.

Add the following:—

(1) $3ax^2+bx+2c$, $2bx^2+ax-3d$, $2cx^2-cx+e$.

(2) $cx^2+4bxy+dy^2$, $2ax^2-3xy+4y^2$, $5fx^2-2hxy+ly^2$.

(3) $2yz+5z^2$, $ayz+bz^2$, $cyz-dz^2$, $eyz+2fz^2$.

(2) Subtraction of Algebraic Expressions.

75. By the rule of brackets (61)

$$(2a+3b-c+d)-(3a-2b+2c-e)$$

is equivalent to

$$2a+3b-c+d-3a+2b-2c+e \text{ or } -a+5b-3c+d+e,$$

whence the Rule:—To subtract one algebraic expression (the subtrahend) from another (the minuend), write the subtrahend after the minuend with all its signs changed and proceed to simplify, by collecting like terms.

It comes to the same thing, if, instead of writing the expressions in order one after the other, we arrange the subtrahend below the

minuend, and then, mentally changing the signs of all the lower terms, *add them.*

$$\begin{array}{r} 2a+3b-\ c+d \\ 3a-2b+2c-e \\ \hline -a+5b+3c+d+e \end{array} \qquad \begin{array}{r} 3a^2+2b^2-\ xy+\ z^2 \\ 2a^2-5b^2-7xy-5z^2 \\ \hline a^2+7b^2+6xy+6z^2 \end{array}$$

Exercises. No. 16.

Subtract:—

(1) $2a-3b$ from $3a-5b$.

(2) $3a-5b+c$ from $4a-2b-3c$.

(3) $2a-5b+6c$ from $a-7b+9c$.

(4) $2ax-5by+6cz$ from $18ax-2by+9cz$.

(5) $7ab^3-3a^2b^2+2ab^3$ from $10ab^3-5a^2b^2+4ab^3$.

(6) $3ax^3-4bx^2y+5cxy^2-7dy^3$ from $2ax^3+12bx^2y-16cxy^2+18dy^3$.

(7) $5x^3-7y^3$ from $13y^3$.

(8) $\frac{x^3}{3}-\frac{2x^2y}{7}+\frac{xy^2}{9}-\frac{y^3}{11}$ from $\frac{x^3}{2}-3x^2y+\frac{2xy^2}{3}-y^3$.

(9) $7x-3y+5$ from 2.

76. When there are unlike literal coefficients, the remainder is classified in subtraction according to the letter of reference, and the literal coefficients are placed in brackets.

Thus,—

$$(ax^2+bx+c)-(dx^2+ex+f)=(a-b)x^2+(b-e)x+c-f.$$

Exercises. No. 17.

Subtract:—

(1) $3bx^2+5ex+6f$ from $2ax^2+3bx+5c$.

(2) $2ax+by+cz+d$ from $3bx+5ay-2d$.

(3) $dx^3+ex^2y+fy^3$ from $ax^3+2ex^2y-gy^3$.

(3) Multiplication of Algebraic Expressions.

77. We have learnt in Arithmetic that if we multiply a number by the sum of two others (or the difference) we obtain the same result as if we multiplied it by these numbers separately and added the products (or took their difference).

In Algebraic language we have $a(c+d) = ac+ad$

$$b(c+d) = bc+bd.$$

Also, $a(c+d) = (c+d)a$, the order of factors being indifferent. If therefore we add the above equalities, we have $(c+d)$ taken a times $+(c+d)$ taken b times or $(c+d)$ taken $(a+b)$ times.

Hence, $(a+b)(c+d) = ac+ad+bc+bd$ (i).

And again, $a(c-d) = ac-ad$

$$b(c-d) = bc-bd.$$

Subtracting, we have $(c-d)$ taken a times less $(c-d)$ taken b times, or $(a-b)(c-d) = ac-ad-bc+bd$ (ii).

We may also reason out any cases, as for example $(a-b)(c+d)$, as follows; $(a-b)$ taken $(c+d)$ times is $(a-b)$ taken c times $+(a-b)$ taken d times $= (a-b)c+(a-b).d$

$$= ac-bc+ad-bd.$$

78. The examination of the several terms of (ii) shows us that the first term $+ac$ results from the concurrence of $+a$, and $+c$ in the factors: the second term $-ad$ from $+a$ and $-d$ in the factors: the third term $-bc$ from $-b$ and $+c$: and the fourth term $+bd$ from $-b$ and $-d$. These results are comprised under what is called the 'Rule of Signs' for multiplication, viz. *Like signs produce* $+$; *unlike signs* $-$.

79. There are difficulties connected with the conception of the meaning of negative signs in multiplication which cannot be fully considered at this time: for the present the fact should suffice that when in any operation the signs $++$, $+-$, $-+$, $--$, concur, as above, the sign in the product follows the Rule of Signs. And if separate terms are given for multiplication, they must be combined according to the same Rule.

Thus, $+2a \times 4b = 8ab$, $+2a^2x \times (-4b^2) = -8a^2b^2x$,

$-2ab \times (+3bc) = -6ab^2c$, $(-3ab^3) \times (-6a^2b) = 18a^3b^4$.

Exercises. No. 18.

Multiply together:—

(1) $+2 \times -3$. (2) -5×-6. (3) $-3a \times 2bc$.

(4) $\frac{1}{2}xy \times -\frac{2x}{3}$. (5) $-3a^2b \times -5bc$. (6) $5a^2bc \times -7a^3b^2c$.

80. The general rule for the multiplication of two expressions will now be:—Multiply each term of the multiplicand by each term of the multiplier in succession, determining the sign of every product by the Rule of Signs; then collect the terms, as in Addition.

It is usual for the beginner to arrange the terms of the multiplier below the multiplicand, as in Arithmetic, placing like terms as far as possible under like and proceeding as in the patterns now given.

81. To multiply $a+b$ by $a+b$.

$$\begin{array}{r} a+b \\ a+b \\ \hline a^2+\ ab \\ \quad\ ab+b^2 \\ \hline a^2+2ab+b^2 \end{array}$$

We first multiply the multiplicand by a, placing each product under the corresponding term of the multiplicand. We then multiply by b arranging the like terms of product under each other. The sum of the partial products gives the complete product. The result is an important one, to be remembered: viz. $(a+b)^2 = a^2+2ab+b^2$, or, in words,—The square of the sum of two numbers is the sum of their squares *together with* twice their product.

82. To multiply $a-b$ by $a-b$.

$$\begin{array}{r} a-b \\ a-b \\ \hline a^2-\ ab \\ \quad -\ ab+b^2 \\ \hline a^2-2ab+b^2 \end{array}$$

We proceed as before, remembering the Rule of Signs.

The result, like the last, is important, and to be noted:—viz. $(a-b)^2 = a^2-2ab+b^2$, or, in words,—The square of the difference of two numbers is the sum of their squares *less* twice their product.

It will be observed that in each case four times the product of the two extreme terms is equal to the square of the middle term.

Exercises. No. 19.

Write down, from the results just obtained, the squares of—

(1) $x+2$. (3) $x+6$. (5) $b+3a$. (7) $7a-3b$.
(2) $x+3$. (4) $a+3b$. (6) $2a+5b$. (8) $x-5b$.
(9) $y-5z$. (10) $3z-3m$.

83. Conversely, whenever we have an expression of the form x^2+6x+9, or x^2-6x+9, where the last term is the square of half the coefficient of x, or, more generally, whenever the square

of the middle term is equal to four times the product of the extreme terms—the expression is the square of a binomial. Thus,

$\sqrt{x^2-8x+16} = x-4$, and $\sqrt{4x^2+36xy+81y^2} = 2x+9y$.

Exercises. No. 20.

Write down the square roots of the expressions:—

(1) $x^2-10x+25$.
(2) $x^2-14x+49$.
(3) $x^2+22x+121$.
(4) $y^2-4xy+4x^2$
(5) $4x^2+20xy+25y^2$.
(6) $49x^2-42xy+9y^2$.

84. To multiply $a+b$ by $a-b$.

The terms $+ab$ and $-ab$ here destroy or cancel each other.

$$\begin{array}{r} a+b \\ a-b \\ \hline a^2+ab \\ -ab-b^2 \\ \hline a^2 \quad -b^2 \end{array}$$

Result. $(a+b)(a-b) = a^2-b^2$, or, in words,—The product of the sum and difference of two numbers is equal to the difference of their squares.

85. The sign $\sim$ is sometimes used to express the difference of a and b when it is not known which is the greater. If we employ this sign, the above result would be expressed

$$(a+b)(a \sim b) = a^2 \sim b^2.$$

This result is of great importance. It enables us, inversely, to resolve any binomial expression, which is the difference of two squares, into two binomial factors.

Examples.—

$x^2-9 = (x+3)(x-3)$, $\quad 4x^2-25 = (2x+5)(2x-5)$,
$x^2-16 = (x+4)(x-4)$, $\quad x^2-a = (x+\sqrt{a})(x-\sqrt{a})$,
$(a+b)^2-c^2 = (a+b+c)(a+b-c)$.

Exercises. No. 21.

Resolve into their factors the expressions:—

(1) x^2-64.
(2) $4y^2-81$.
(3) $36-25x^2$.
(4) $9-(3y+2x)^2$.
(5) $16-(4-3y)^2$.
(6) $25-(2y-3z)^2$.
(7) $(a-b)^2-c^2$.
(8) $(y+z-x)^2-(x+y-z)^2$.

86. To multiply $x+a$ by $x+b$.

The result is an expression in descending powers of x, the

first term being x^2, the second x multiplied by the sum of a and b, and the last term the product of a and b.

$$\begin{array}{r} x+a \\ x+b \\ \hline x^2+ax \\ \quad bx+ab \\ \hline x^2+(a+b)x+ab \end{array}$$

Exercises. No. 22.

Find, by actual multiplication, the products of the factors—

(1) $(x+2)(x+3)$.
(2) $(a+b)(a+c)$.
(3) $(a+3)(a+7)$.
(4) $(x+5)(x+9)$.
(5) $(y+9)(y+2)$.
(6) $(2x+3)(2x+5)$.
(7) $(3x+2y)(3x+5y)$.
(8) $(5x+2z)(5x+3z)$.
(9) $(3z+2x)(5z+2x)$.
(10) $(7x+2y)(3x+2y)$.

Exercises. No. 23.

Write down, without actual multiplication, from the form of the result in (86), the products of—

(1) $(a+3)(a+5)$.
(2) $(x+1)(x+2)$.
(3) $(2x+1)(2x+3)$.
(4) $(3x+y)(3x+4y)$.

87. The equality $(x+a)(x+b)=x^2+(a+b)x+ab$ will apply to cases in which a or b is negative, or both of them.

We may write $-a$ for a, $-b$ for b; and we thus obtain—

$$(x+a)(x-b)=x^2+(a-b)x-ab,$$
$$(x-a)(x+b)=x^2-(a-b)x-ab,$$
$$(x-a)(x-b)=x^2-(a+b)x+ab.$$

Exercises. No. 24.

Write down, without multiplication, from the forms of the results in (87) the products of—

(1) $(x+3)(x-7)$.
(2) $(a+3b)(a-5b)$.
(3) $(x-3y)(x-5y)$.
(4) $(b+5c)(b-7c)$.
(5) $(3+y)(7-y)$.
(6) $(3z+x)(5z-x)$.

88. Conversely, we can often resolve trinomial expressions, such as x^2+3x+2, into binomial factors. We have only to find two factors of the last term, such that their algebraical sum is the coefficient of x; these will be the last terms of the binomial factors. We must remember that the algebraical factors of quantities are the arithmetical, affected with either + or −, and therefore twice as numerous. Thus the factors of $+2$ and -2 are equally $+1, -1, +2, -2$, properly combined, since

$1\times 2 = 2$, $-1\times -2 = 2$, $1\times -2 = -2$, and $-1\times 2 = 2$.

Since the positive factors of 2, 1 and 2 have their sum = 3,

$$\therefore\ x^2+3x+2 = (x+1)(x+2).$$

But if we have the expression x^2-3x+2, the positive factors of 2, 1 and 2 will no longer serve, since their algebraical sum is not -3. But $(-1)\times(-2) = 2$, and their algebraical sum is -3,

$$\therefore\ x^2-3x+2 = (x-1)(x-2).$$

Again, $+3, -5$ answer the conditions for $x^2-2x-15$; and $-3, +5$ for $x^2+2x-15$.

Exercises. No. 25.

Resolve the following expressions into their factors:—

(1) $x^2+8x+15$.
(2) $x^2+10x+21$.
(3) $x^2-2x-24$.
(4) $x^2-20x-21$.
(5) $x^2+14x-51$.
(6) x^2-x-42.
(7) $x^2-12xy-13y^2$.
(8) $x^2+10xy-11y^2$.

89. It is not always that a trinomial admits of being resolved into factors with integers for their last terms. The following supplies a certain method of discovering the factors, of whatever form, when they exist: but it is longer than the foregoing.

Since $(x+a)^2 = x^2+2ax+a^2$, we may see that any binomial of the form x^2+2ax may be made a complete square by the addition of the square of half the coefficient of x. This is a result which will be of great use to us hereafter. It follows that a trinomial of the form $x^2+2bx+c^2 = (x+b)^2-(b^2-c^2)$.

And if b^2-c^2 be positive, i.e. if $b>c$, we may make use of brackets to put it in the form

$$x^2+2bx+c^2 = \{x+b+\sqrt{b^2-c^2}\}\{x+b-\sqrt{b^2-c^2}\}.$$

Examples. $x^2+8x+12=(x+4)^2-4=(x+6)(x+2)$

$x^2-7x+\frac{45}{4}=(x-\frac{5}{2})(x-\frac{9}{2})$.

90. When expressions to be multiplied together contain several terms, they should be arranged, when possible, according to the powers of some one letter, all in ascending or descending order, as in the annexed example. The result may be noted, as enabling us in some cases to resolve an expression into factors.

$$\begin{array}{l} a^2+ab+b^2 \\ a^2-ab+b^2 \\ \hline a^4+a^3b+a^2b^2 \\ \quad -a^3b-a^2b^2-ab^3 \\ \qquad +a^2b^2+ab^3+b^4 \\ \hline a^4 \qquad +a^2b^2 \qquad +b^4 \end{array}$$

Additional Examples.

$$\begin{array}{l} x^3-3x^2+2x+1 \\ 2x^2-5x+4 \\ \hline 2x^5-6x^4+4x^3+2x^2 \\ \quad -5x^4+15x^3-10x^2-5x \\ \qquad 4x^3-12x^2+8x+4 \\ \hline 2x^5-11x^4+23x^3-20x^2+3x+4 \end{array}$$

$$\begin{array}{l} x^3+ax^2+b \\ x^2+cx+d \\ \hline x^5+ax^4+bx^2 \\ \quad cx^4+acx^3+bcx \\ \qquad dx^3+adx^2+bd \\ \hline x^5+(a+c)x^4+(ac+d)x^3 \\ \qquad +(b+ad)x^2+bcx+bd \end{array}$$

Exercises. No. 26.

Multiply:—

(1) $8xy-3y+z$ by $2xy+y-5z$.

(2) $7x^2y-5xy^2+6y^3$ by $2x^3-3x^2y-5y^3$.

(3) $x^4+x^3+x^2+x+1$ by $x-1$.

(4) $x^6+x^4+x^2+1$ by $x^6-x^4+x^2-1$.

(5) $abc-3bc^2+4c^3$ by $3a^3-2a^2b+4b^3$.

(6) $7a^3+6a^2b-5b^3$ by $6a^2b-2ab^2+7b^3$.

(7) $8y^3-3y^2+2y-5$ by $7y^2-8y+9$.

(8) $1+\frac{x}{2}+\frac{x^2}{4}$ by $1-\frac{x}{2}+\frac{x^2}{8}-\frac{x^3}{16}$.

(9) $(x^2-xy+y^2)^2$ by $(x+y)^2$.

(10) $(a^2+ab+b^2)^2$ by $(a-b)^2$.

(11) $(7l-2m-9)\times(3l-11m)$.

(12) $(3a+5b-\frac{7c}{2})\times(a-2b+9c)$.

(13) $(5a^2-3ab+7b^2)\times(3a-b)$.

(14) $(3a^3-5a^2b-\frac{1}{2}ab^2)\times(-2a^2-3ab+b^2)$.

(15) $(a^2+2ab+b^2-c^2)(a^2+2ab-b^2+c^2)$.

(16) $(a^2-2a+4)(a^2+2a+4)$.

(17) $(a^2+2ab-2b^2)(a^2+2ab+2b^2)$.

(18) $a^2+b^2+c^2-a(b+c)-bc$ by $a+b+c$.

(19) $a^2+b^2+c^2+d^2-a(b+c+d)-b(c+d)-cd$ by $a+b+c+d$.

(20) $(14a^5c^2-6a^2bc^2+c^3)(14a^5c^2+6a^2bc^2-c^3)$.

To find the continued product of

$$(x-1)\times(x+2)\times(x+3).$$

We first multiply $x-1$ by $x+2$, and afterwards multiply the product by $x+3$. The result may be proved by taking the factors in a different order.

$$\begin{array}{r} x-1 \\ x+2 \\ \hline x^2-x \\ \quad 2x-2 \\ \hline x^2+x-2 \\ x+3 \\ \hline x^3+x^2-2x \\ \quad 3x^2+3x-6 \\ \hline x^3+4x^2+x-6 \end{array}$$

Find the continued product of:—

(21) $(x+a)(x+b)(x+c)$.

(22) $(x+a)(x-b)(x-c)$.

(23) $(x+3)(x-2)(x+5)$.

(24) $(x+2)(x+2)(x+2)$ or $(x+2)^3$.

(25) $(x-1)^3$. (26) $(a+b+c)^3$. (27) $(a-b-c)^3$.

(28) $(l+\frac{1}{l})(l^2+\frac{1}{l^2})(l-\frac{1}{l})$.

(29) $(a+b+c)(a+b-c)(a-b+c)(-a+b+c)$.

(30) $(a+b+c+d)(a-b+c-d)(a+b-c+d)$.

(31) $(m+n)(n+p)(p+q)$.

(32) $(m-n)(m+n)(m^2+n^2)(m^4+n^4)$.
(33) $(a^3+a-1)(a^2-a+1)(a^2+a+1)$.
(34) $(p^2+q^2+4r^2)(p^2-q^2+4r^2)(p^2+q^2-4r^2)$.

(4) Division of Algebraic Expressions.

91. Division, as in Arithmetic, is an operation by which we find how often one of two given quantities (the dividend) contains the other (called the divisor). The result is called the quotient.

92. We have already treated of the division of single terms, except that we have not employed the sign $-$. But, division being the inverse of Multiplication, the rule of signs follows at once from the investigations in Multiplication.

$$\because (+a)\times(+b) = +ab, \qquad \therefore \frac{+ab}{+a} = +b;$$

$$\because (+a)\times(-b) = -ab, \qquad \therefore \frac{-ab}{+a} = -b;$$

$$\because (-a)\times(+b) = -ab, \qquad \therefore \frac{-ab}{-a} = +b;$$

$$\because (-a)\times(-b) = +ab, \qquad \therefore \frac{+ab}{-a} = -b;$$

i.e. in Division as in Multiplication, like signs produce $+$; unlike signs $-$.

93. **Short Division.** If we have to divide an algebraical expression by a single term, we arrange the quantities as in Short Division in Arithmetic. The operation consists in dealing with the successive terms separately, as in the division of single terms, and in writing down the quotients with their appropriate signs.

$$b \,)\, 3a^2b+5ab^2-6b^3$$
$$3a^2+5ab-6b^2$$

Exercises. No. 27.

Divide :—

(1) $3x^2y-15xy^2+6y^3$ by $3y$.
(2) $81a^3bc-42a^2b^2c+27abc^3$ by $-9ac$.

Find the values of—

(3) $\dfrac{5xyz-100yz^2+20z^3}{5z}$. (4) $\dfrac{91x^3z-51xz^3+39z^4}{-3z}$.

94. **Long Division.** The arrangement is similar to that in Arithmetic.

Division being the inverse or undoing of a multiplication, we will exhibit the two processes in connection with each other.

$$
\begin{array}{l}
 3a+5b-2c \\
 2a-3b+c \\
\hline
 6a^2+10ab-4ac \\
 \quad -9ab \qquad -15b^2+6bc \\
 \qquad\quad 3ac \qquad\quad +5bc-2c^2 \\
\hline
3a+5b-2c\ )\ 6a^2+\ ab-\ ac-15b^2+11bc-2c^2\ (\ 2a-3b+c \\
\quad 6a^2+10ab-4ac \\
\quad \hline
\quad -9ab+3ac-15b^2+11bc \\
\quad -9ab- \qquad 15b^2+6bc \\
\quad \hline
\qquad\quad 3ac \qquad +5bc-2c^2 \\
\qquad\quad 3ac \qquad +5bc-2c^2 \\
\qquad\quad \hline
\qquad\quad * \qquad\quad * \quad * \\
\qquad\quad \hline\hline
\end{array}
$$

Suppose that we have multiplied $3a+5b-2c$ by $2a-3b+c$, and are now returning to the original multiplier by treating the product as dividend, and making the multiplicand our divisor. We divide the first term of the dividend by the first term of the divisor, and find the quotient $2a$. Multiplying the divisor by $2a$, we form $6a^2+10ab-4ac$, which answers to the first partial product in the multiplication. We withdraw this and form successively the quotients -3, and $+c$. There is no remainder, which shows that the dividend is divided exactly by the divisor $3a+5b-2c$.

In the next examples observe that we carefully arrange both divisor and dividend according to descending powers of the same letter, x. Ascending powers will do, if we prefer it, and in fact the second example may be said to be arranged according to

ascending powers of y: but one of these orders must be followed, and the same in both divisor and dividend.

$$\begin{array}{l|l} 2x+1 & 2x^2-x-1 \\ & 2x^2+x \\ \hline & -2x-1 \\ & -2x-1 \\ \hline & \quad * \quad * \\ \hline\hline \end{array} \qquad x-1$$

$$\begin{array}{l|l} 2x+y & 2x^2-xy-y^2 \\ & 2x^2+xy \\ \hline & -2xy-y^2 \\ & -2xy-y^2 \\ \hline & \quad * \quad * \\ \hline\hline \end{array} \qquad x-y$$

We subjoin some further examples:—

$$\begin{array}{l|l} 1+x+x^2 & 1-x^3 \\ & 1+x+x^2 \\ \hline & -x-x^2-x^3 \\ & -x-x^2-x^3 \\ \hline & \quad * \quad * \quad * \\ \hline\hline \end{array} \qquad 1-x$$

$$\begin{array}{l|l} x+y & x^4-y^4 \\ & x^4+x^3y \\ \hline & -x^3y-y^4 \\ & -x^3y-x^2y^2 \\ \hline & \qquad x^2y^2-y^4 \\ & \qquad x^2y^2+xy^3 \\ \hline & \qquad -xy^3-y^4 \\ & \qquad -xy^3-y^4 \\ \hline & \qquad\quad * \quad * \\ \hline\hline \end{array} \qquad x^3-x^2y+xy^2-y^3$$

$$\begin{array}{l|l} 3a+5b-2c & 12a^2+11ab-8ac-15b^2+6bc \\ & 12a^2+2ab-8ac \\ \hline & \quad -9ab \quad -15b^2+6bc \\ & \quad -9ab \quad -15b^2+6bc \\ \hline & \quad * \quad\quad * \quad * \\ \hline\hline \end{array} \qquad 4a-3b$$

Exercises. No. 28.

Divide:—

(1) $x^3-5x^2y+8xy^2-4y^3$ by $x-2y$.

(2) $12x^4+x^3-10x^2+4x-3$ by $3x^2+x-3$.

(3) $30x^6+5x^5-x^4+5x^3+x$ by $5x^3-x+1$.

(4) $b-3b^2+3b^3-b^4$ by $b-1$.

(5) $k^{10}-k$ by k^3-1.

(6) $6x^6+2x^4-x^3-18x^2-22x+35$ by $2x^3+3x-7$.

(7) x^6+64 by x^2+4.

(8) $x^5+3x^4-6x^3-13x^2+5x-10$ by x^3+3x^2-x+2.

(9) $x^5-5x^4+4x^3+10x^2-33x+7$ by x^3-3x^2+5x-1.

(10) $a^8-3a^7-9a^6-5a^5+13a^2+51a+18$ by a^2-3a-9.

(11) x^5+x^4-x-1 by x^3+x^2+x+1.

(12) $a^6+4a^5+4a^4-16a^3-64a-64$ by a^3-2a^2+4a-8.

(13) $y^4-9y^2z^2+6yz^3-z^4$ by $y^2+3yz-z^2$.

(14) $x^2+7\sqrt{x}-12$ by $x+\sqrt{x}-3$.

(15) $x^6-2x^3+17x\sqrt{x}-72$ by $x^3+x\sqrt{x}-9$.

(16) $a^8-5a^6+7a^5+6a^4-20a^3+10a^2+2a-5$ by a^3-2a+5.

(17) x^6+64 by $x-2\sqrt{x}+2$.

(18) Show by actual division that

$$(x^6-64)\div(x^3-8)=(x^6-64)\div(x+2)(x^2-2x+4).$$

(19) Divide x^8-256 successively by $x-2$, $x+2$, x^2+4.

(20) $x^5+6x^4+4x^3-4x^2+x-1$ by x^3+5x^2+1.

(21) $2a^4-13a^3+31a^2-38a+24$ by a^2-5a+6.

(22) $6a^6-13a^5+28a^4-23a^3+20a^2$ by $2a^2-3a+4$.

(23) $8m^5+22m^4-5m^3-46m^2-17m+10$ by $2m^3+3m^2-4m-5$.

(24) $20p^5-41p^4+50p^3-45p^2+25p-6$ by $5p^3-4p^2+5p-3$.

(25) $14l^5-27l^4+21l^3-3l^2-2l$ by $2l^2-3l+2$.

(26) x^4+1 by $x^2+x\sqrt{2}+1$.

(27) $18+9a-53a^2-9a^3$ by $3+a-9a^2$.

(28) $x^6-6x^5+21x^4-44x^3+63x^2-54x+27$ by x^2-2x+3.

(29) $\frac{1}{8}a^4+\frac{1}{2}a^3-\frac{7}{9}a^2+\frac{1}{3}a-\frac{2}{8}$ by $\frac{1}{3}a^2+\frac{1}{2}a-1$.

(30) $3x^3-3$ by $\frac{1}{2}x^2+\frac{1}{2}x+\frac{1}{2}$.

(31) $y^5-\frac{1}{243}$ by $y-\frac{1}{3}$.

(32) $x^4-\frac{1}{81}$ by $x^3+\frac{2}{3}x^2+\frac{4}{9}x+\frac{8}{27}$.

(33) $2-\frac{1}{8}x^4$ by $2+x$.

(34) $3-\frac{x^6}{243}$ by $3-x$.

(35) $1-5x+10x^2-10x^3+5x^4-x^5$ by $1-2x+x^2$.

(36) x^8+x^4+1 successively by x^2+x+1 and x^2-x+1.

(37) $x^6-5x^5+14x^4-28x^3+39x^2-39x+18$ successively by x^2-3x+2 and x^2-2x+3.

95. Hitherto we have had an exact quotient, but this is not always the case. Suppose we are required to find $3x \div (2x-a)$, or $4x-b \div (7x-a)$. In neither of these cases can we do more than write $\frac{3x}{2x-a}$, $\frac{4x-b}{7x-a}$, fractions which are considered to be the quotients, the results of division. It will often also happen in Short or in Long Division that after obtaining a certain number of terms we come at last to a term or terms which can only be treated as above, and their quotient exhibited as a fraction. For example :—

$$\frac{3a^3x^2+5ax^4+6bc^4}{ax} = 3a^2x+5x^3+\frac{6bc^4}{ax}.$$

Such a result as the above answers to a mixed number in Arithmetic, partly integral and partly fractional, and accordingly the part of the quotient $3a^2x+5x^3$ which precedes the fraction is said to be 'an integral expression.' Even if the numerical coefficients should be fractional, as in $\frac{1}{3}a^2x+\frac{1}{8}x^3$, it would still be said to be 'an integral expression with respect to x,' for x does not appear in the denominator. See also (120) below.

Again, required $\frac{x^3+y^3}{x-y}$.

$$\begin{array}{r|l} x-y & x^3+y^3 \left(x^2+xy+y^2+\frac{2y^3}{x-y} \right. \\ & \underline{x^3-x^2y} \\ & \quad x^2y+y^3 \\ & \quad \underline{x^2y-xy^2} \\ & \qquad xy^2+y^3 \\ & \qquad \underline{xy^2-y^3} \\ & \qquad\quad 2y^3 \end{array}$$

The last remainder may be removed by using for quotient the fraction $\frac{2y^3}{x-y}$, and the quotient must include this fraction to be complete.

96. By observing the successive remainders in this division, and in that of $(x^4-y^4) \div (x+y)$ on page 37, we may perceive what

the cases are in which the divisions indicated below will give exact quotients. See also the method of (98), below.

	Form of Quotient.	Remainder.	When divisible.	Examples.
$\frac{x^n+a^n}{x-a}$	$x^{n-1}+ax^{n-2}+\dots$	$2a^n$	never	$\frac{x+a}{x-a}$ $\frac{x^2+a^2}{x-a}$
$\frac{x^n+a^n}{x+a}$	$x^{n-1}-ax^{n-2}+\dots$	$(-a)^n+a^n$	n odd	$\frac{x+a}{x+a}$ $\frac{x^2+a^2}{x+a}$
$\frac{x^n-a^n}{x-a}$	$x^{n-1}+ax^{n-1}+\dots$	0	always	$\frac{x-a}{x-a}$ $\frac{x^2-a^2}{x-a}$
$\frac{x^n-a^n}{x+a}$	$x^{n-1}-ax^{n-2}+\dots$	$(-a)^n-a^n$	n even	$\frac{x-a}{x+a}$ $\frac{x^2-a^2}{x+a}$

The last column contains the fractions for $n=1$, $n=2$, the inspection of which answers the question, whenever we are in doubt, since all odd cases come under the same rule, and all even. For example:—

$$\frac{x^3+a^3}{x+a}=x^2-ax+a^2, \qquad \frac{x^2-a^2}{x-a}=x+a, \qquad \frac{x^3-a^3}{x-a}=x^2+ax+a^2.$$

The alternation of signs in the quotient happens *where it does not in the divisor* and vice versâ.

Exercises. No. 29.

Write down the quotients and remainders (if any) in the following cases:—

(1) $\frac{x^4+a^4}{x-a}$. (4) $\frac{x^5-y^5}{x+y}$. (7) $\frac{x^3-8}{x-2}$.

(2) $\frac{x^4+a^4}{x+a}$. (5) $\frac{x^6+y^6}{x-y}$. (8) $\frac{x^4-16}{x+2}$.

(3) $\frac{x^5-y^5}{x-y}$. (6) $\frac{x^9-y^9}{x+y}$. (9) $\frac{x^5-243}{x-3}$. (10) $\frac{81-y^7}{3+y}$.

97. We give some further examples with literal coefficients:—

$$\begin{array}{r|l} 2a+3x & 4a^2+6ab-4ax+9bx-15x^2 \\ & 4a^2+6ab+6ax+9bx \\ \hline & -10ax-15x^2 \\ & -10ax-15x^2 \\ \hline \end{array} \qquad (2a+3b)-5x$$

Exercises. No. 30.

Divide :—

(1) $a^4+a^2b^2+b^4$ by a^2-ab+b^2.

(2) $a^8+a^4b^4+b^8$ by a^2+ab+b^2.

(3) $ab(x^2+y^2)+xy(a^2+b^2)$ by $ax+by$.

(4) $a^2-b^2-c^2+d^2-2(ad-bc)$ by $a+b-c-d$.

(5) $\frac{x}{a}-1-\frac{b}{a}-\frac{b^2}{a^2}+\frac{b}{x}+\frac{b^2}{x^2}$ by $x-a$.

(6) $x^6-na^2x^4+na^4x^2-a^6$ by x^2-a^2.

(7) $x^4+(2b^2-a^2)x^2+b^4$ by x^2+ax+b^2.

(8) $mqx^4+(nq+mr)x^3+nrx^2+pqx+pr$ by $qx+r$.

(9) $4c^4-9b^2c^2+6b^3c-b^4$ by $2c^2-3bc+b^2$.

(10) $\frac{3}{4}x^5-4x^4+\frac{7}{8}x^3-\frac{4}{4}\frac{3}{3}x^2-\frac{3}{4}\frac{3}{3}x+27$ by $\frac{1}{2}x^2-x+3$.

(11) $-1+a^3n^3$ by $-1+an$.

(12) $a^6+2a^3z^3+z^6$ by a^2-az+z^2.

(13) $72x^4-78x^3y-10x^2y^2+17xy^3+3y^4$ by $6x^2-4xy-y^2$.

(14) $a^4-9a^2b^2-6abc^2-c^4$ by $a^2-3ab-c^2$.

(15) $32a^5+b^5$ by $2a+b$.

(16) $2a^4-13a^3b+31a^2b^2-38ab^3+24b^4$ by $2a^2-3ab+4b^2$.

(17) $4a^2+(6b-4x)a+9bx-15x^2$ by $2a+3b-5x$.

(18) Multiply $(a-2)(a-1)a(a+1)(a+2)$, and divide the product by a^2-a-2.

(19) Multiply a^3-3a+2 by a^3-3a^2+2, and divide the product by a^2-2a+1.

(20) Divide $acr^3+(bc+ad)r^2+(bd+ac)r+bc$ by $ar+b$, and test the result by putting $a=b=c=d=e=1$, and $r=10$.

98. The following example has a great importance :—

$$
\begin{array}{l|l}
x-a & x^3+px^2+qx+r \;\big(\; x^2+(a+p)x+(a^2+pa+q) \\
 & \underline{x^3-ax^2} \\
 & \quad (a+p)x^2+qx \\
 & \quad \underline{(a+p)x^2-(a^2+pa)x} \\
 & \qquad (a^2+pa+q)x+r \\
 & \qquad \underline{(a^2+pa+q)x-(a^3+pa^2+qa)} \\
 & \qquad\qquad a^3+pa^2+qa+r
\end{array}
$$

The remainder in this case is exactly what the dividend becomes if we write in it a for x, and it will be observed that if we write down the coefficients of the powers of x in order in separate columns, and using a as a multiplier, carry the result to be added to the next column, multiply that result by a and carry it to the next column, and so on, the last result will be the above remainder, which is the value of the expression when x is made equal to a, and the other columns contain the coefficients of the quotient. This is the first step in what is called Horner's process, and it furnishes the easiest way (i) of dividing any expression in powers of x by a binomial of the form $x-a$, (ii) of finding the value of such an expression when a particular value, as a, is given to x; (iii) of ascertaining whether the expression is exactly divisible by $x-a$, or, what is the same thing, whether there is any remainder. The following is the work indicated.

1	p	q	r
	$a+p$	a^2+pa+q	$a^3+pa^2+qa+r.$

In writing down the heads of the columns, it is necessary to supply 0 as the coefficient of every power of x which is wanting.

Example. Divide x^6-1 by $x-1$.

$x^6+0.x^5+0.x^4+0.x^3+0.x^2+0.x$	-1
1 0 0 0 0 0	-1
1 1 1 1 1	0

Quotient $x^5+x^4+x^3+x^2+x+1$.

Remainder $= 0 =$ value x^6-1 when $x = 1$.

Example. Divide x^3-a^3 by $x+a$. Here, as we have to change the sign of the last term, $-a$ is the operator:

1	0	0	$-a^3$,
	$-a$	$+a^2$	$-2a^3$.

Quotient x^2-ax+a^2.

Remainder $= -2a^3 =$ value of x^3-a^3 when $x = -a$.

Example. Divide $x^5+3x^3+2x-15$ by $x-3$.

1	0	3	0	2	-15
	3	12	36	110	315

Quotient $x^4+3x^3+12x^2+36x+110$.

Remainder $= 315 =$ value of the expression when $x = 3$.

Exercises. No. 31.

Divide, giving remainders, if any :—

(1) $x^4+3x^3+2x+15$ by $x-3$, and $x-1$.

(2) $3x^6+2x^4-7$ by $x-5$, $x-2$, and $x+2$.

(3) $7x^8-2x^3+5$ by $x+1$, $x-2$, and $x+2$.

(4) $3x^5+2x^4y+3x^2y^3+6y^5$ by $x-2y$.

CHAPTER V.

ALGEBRAIC FRACTIONS.

(1) General Notion of Fractions.

99. The theory of fractions in Algebra rests upon that of fractions in Arithmetic, and the operations upon algebraic fractions are in most respects in such complete accordance with those in Arithmetic that it would be practically sufficient, if we can assume a knowledge of Arithmetic, to say :—Treat algebraical fractions as if they were arithmetical. We propose therefore to give the general theory briefly, and to dwell only on the points of difference between algebraical and arithmetical fractions.

100. The simplest form of a fraction is $\frac{a}{b}$, where a and b are integers. The denominator b shows that the primary or original unit, which must be conceived as concrete (See *Scholar's Arithmetic*, V. 1), is divided into b subordinate units, and the numerator shows that a of these are taken. Thus a is really the number, and b shows the kind of unit in which a is reckoned.

101. We cannot speak of proper and improper fractions in the same sense as in Arithmetic, with reference to the relative magnitude of a and b: but we use these names in reference to form. Thus, a not being divisible by b, $\frac{a}{b}$ is a proper fraction, and $\frac{bx+a}{b}$, having a numerator which is algebraically divisible by b,

is an improper fraction, equivalent to the mixed quantity $x+\frac{a}{b}$ and the mixed quantity $cx+\frac{a}{b}=\frac{bcx+a}{b}$.

Here we have one of the important differences in notation between arithmetical and algebraical fractions. In Arithmetic we write $1\frac{2}{3}$, $2\frac{1}{3}$, $3\frac{1}{4}$ as mixed quantities, which signify $1+\frac{2}{3}$, $2+\frac{1}{3}$, $3+\frac{1}{4}$. We cannot thus omit the + in $cx+\frac{a}{b}$, for $cx\,\frac{a}{b}$ would be taken to mean $cx.\frac{a}{b}=\frac{acx}{b}$.

As a general rule, when the numerator is an expression of lower dimensions than the denominator, the fraction may be considered 'proper,' if of equal or higher dimensions, 'improper,' because on division we may obtain an integral quotient and a remainder of lower dimensions.

102. We return to the fundamental meaning of $\frac{a}{b}$.

To multiply the numerator a by m, is to take m times as many of these units, or to multiply the fraction by m. Thus, $m\times\frac{a}{b}=\frac{ma}{b}$.

103. To multiply the denominator by m is to divide the unit into m times as many subordinate units. $\frac{a}{mb}$ means that a of these are taken, as before: but every such subordinate unit being so much smaller than before that m of them are required to make up one of the former, it follows that it would take m of the fraction $\frac{a}{mb}$ to make up the former fraction $\frac{a}{b}$. In other words, $\frac{a}{b}$ has been divided by m, $\therefore \frac{a}{b}\div m=\frac{a}{mb}$.

104. Inversely, to pass from $\frac{a}{mb}$ to $\frac{a}{b}$, which we do by dividing the denominator by m, is to take a units m times as large as before, or the fraction is m times as large, and to pass from $\frac{ma}{b}$ to $\frac{a}{b}$ by dividing the numerator by m is to divide the fraction by m, since we take only $\frac{1}{m}$th of the number of units we took before. Thus a

fraction is multiplied by any number, either by multiplying the numerator, or by dividing the denominator by that number, and a fraction is divided by any number, either by multiplying the denominator, or by dividing the numerator, by that number.

105. To multiply both numerator and denominator by m, as $\frac{ma}{mb}$, is to divide every subordinate unit (b of which make the primary unit) into m parts, and to take ma of the mb parts resulting, which is in fact just the same as a of the b parts, for every one of the old divisions contains m of the new. Thus, $\frac{ma}{mb}$ expresses the same magnitude as $\frac{a}{b}$, or $\frac{ma}{mb}=\frac{a}{b}$. (See *Scholar's Arithmetic*, VI. 1.) Thus, to introduce a common factor into the numerator and denominator does not alter its value. Inversely, to remove a common factor from the numerator and denominator of a fraction does not alter its value.

106. A fraction is said to be in its lowest terms when all such common factors, numerical or algebraical, have been removed. In speaking of factors, we mean in Algebra those which are apparent in the form of the expression, and we make no reference to those which may be contained in the numerical values of symbols, known or unknown.

Exercises. No. 32.

Put in the form of mixed numbers the improper fractions :-

(1) $\frac{3x+a}{3}$. (3) $\frac{3ax+2by+c}{2b}$. (5) $\frac{a^2-b^2}{b}$.

(2) $\frac{bx+3a}{b}$. (4) $\frac{a^2-b^2}{a}$. (6) $\frac{7a^3-3a^2b+2b^3}{b}$.

Put in the form of improper fractions the mixed numbers :—

(7) $3a+\frac{2b}{c}$. (9) $3ax-by+\frac{2}{y}$. (10) $-7ax+8by-\frac{cy}{x}$.

(8) $3ax-2by+\frac{c}{d}$. (11) $8b^2-\frac{3a^2-2ab}{x}$.

Multiply:—

(12) $\frac{2a}{b^2}$ by $3b$. (14) $\frac{axy}{3z}$ by $2z$.

(13) $-\frac{4b}{7}$ by $3c$. (15) $\frac{4a+3b}{2b^3}$ by $3a$.

Divide:—

(16) $\frac{4b}{2c}$ by $2d$. (18) $\frac{5bxy}{2z}$ by $3y$.

(17) $-\frac{6a^2d}{5c}$ by $3a$. (19) $\frac{3a+b}{2b^3}$ by $2b$.

Simplify:—

(20) $\frac{5ac^2}{17d} \times \frac{1}{3c}$. (23) $\frac{7ax+y}{2c} \times \frac{1}{x}$.

(21) $-\frac{8xy^2}{5ax} \div 2y$. (24) $\frac{3abc}{8x^2y^2} \times \frac{4x}{b^2} \times \frac{1}{2c}$.

(22) $\frac{9y^2z}{5x} \times \frac{1}{3c}$. (25) $\frac{8axy}{3 \times 5c^3} \times 5byz \times 16czx$.

Reduce to lowest terms:—

(26) $\frac{a^2x^2y^3}{bxy^4z}$. (27) $-\frac{3bxyz^3}{27y^2z^2}$. (28) $\frac{8a^3xyz}{4ab^2x^2}$. (29) $-\frac{3bx^3z}{2axy^2z^2}$.

(2) Addition and Subtraction of Fractions.

107. To add two or more fractions is to collect them into one expression with their proper sign. This is done by reducing them to equivalent fractions with a common denominator. We may do this by multiplying the numerator and denominator of every fraction by all the denominators except its own. All this is exactly the same as in Arithmetic. Thus—

$$\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{bc}{bd}=\frac{ad+bc}{bd}.$$

108. For subtraction, having reduced the fractions to a common denominator, we have to subtract the numerators. Thus—

$$\frac{a}{b}-\frac{c}{d}=\frac{ad}{bd}-\frac{bc}{bd}=\frac{ad-bc}{bd}.$$

This is the same thing as to add together $+\frac{a}{b}$ and $-\frac{c}{d}$.

Exercises. No. 33.

Add together:—

(1) $\frac{a}{3}+\frac{b}{4}$. (4) $\frac{a}{3b}+\frac{3c}{d}$. (7) $\frac{a}{3}-\frac{b}{4}$.

(2) $\frac{3}{a}+\frac{4}{b}$. (5) $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}$. (8) $\frac{3}{a}-\frac{4}{b}$.

(3) $\frac{a}{2b}+\frac{2b}{a}$. (6) $\frac{2x}{a}+\frac{3y}{b}+\frac{5z}{c}$. (9) $\frac{10}{x}-\frac{5}{y}+\frac{2}{z}$.

Subtract:—

(10) $\frac{3}{z}$ from $\frac{5}{y}$. (12) $\frac{9c}{b}$ from $\frac{2z}{c}+\frac{2y}{b}$.

(11) $\frac{15a}{b^2}$ from $\frac{7b}{c^2}$. (13) $\frac{a}{3y}-\frac{b}{5z}$ from $\frac{6c}{x}$.

(3) Multiplication and Division of Fractions.

109. To multiply $\frac{a}{b}\times\frac{c}{d}$, multiply the numerators together, as in Arithmetic, for a new numerator and the denominators for a new denominator. Thus, $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$.

The first notion of such multiplying is derived from the case where a, b, c, d are integers and the unit is conceived as concrete. (See *Scholar's Arithmetic*, VI. 9.) This unit being divided into d parts, each part is $\frac{1}{d}$, and c being taken we have a quantity denoted by $\frac{c}{d}$. That which has been done to the unit in constituting the fraction $\frac{c}{d}$, is now to be done to $\frac{c}{d}$, conceived as a concrete magnitude. It is divided into b parts and a of them taken. Each such part is $\frac{1}{bd}$ and ac of them are taken; thus the new fraction is $\frac{ac}{bd}$.

110. Division of Fractions is the inverse process and asks how often $\frac{c}{d}$ is contained in $\frac{a}{b}$. It must be d times as often as c is,

which is $\frac{a}{bc}$. It is therefore $\frac{ad}{bc}$, whence the rule;—Invert the divisor and proceed as in Multiplication:—

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}.$$

111. Although the introduction of fractions is due to the endeavour to represent quantities, yet we gradually throw off all consideration of quantities, and learn to regard them as pure numbers, having properties as much independent of concrete magnitudes as those of integers. In the same manner rules about the fractions of Algebra become, when we can no longer interpret them, by reason of their complexity, rules about the forms of results, such forms being ever those which, when the quantities have numerical values assigned to them, conform to the laws of Arithmetic.

It is only by reference to first principles that the rules for the multiplication and division of fractions can be proved. Indeed this remark is applicable to all mathematical investigation: if there is an obscurity, a reference to first principles will generally remove it.

112. We may now extend the above rules to fractional values of the numerators and denominators.

Let $a = \frac{m}{n}$, and $b = \frac{p}{q}$ where m, n, p, q are prime integers.

Then $\frac{a}{b} = \frac{\frac{m}{n}}{\frac{p}{q}} = \frac{mq}{np}$, by the rule for division of fractions.

Thus, all such fractions can be reduced to fractions with integral numerators and denominators. Negative quantities in the numerator or denominator come under the rule of signs in division (92).

Exercises. No. 34.

Multiply together:—

(1) $\frac{a^3bcxy^2}{dez} \times \frac{ad^2e^2}{3bc^2x^2y}$.

(2) $(\frac{a}{x}+\frac{b}{y}) \times (\frac{x}{b}+\frac{y}{a})$.

(3) $(\frac{1}{a}+\frac{1}{x}) \times (\frac{1}{a^2}-\frac{1}{x^2})$.

(4) $\left\{\frac{1}{(a-b)^2}-\frac{1}{c^2}\right\} \times \frac{a^2-b^2}{a-b-c}$.

Divide :—

(5) $\frac{mnpx^3}{aby^3}$ by $\frac{m^2xy}{a^2c^2}$.

(7) $\frac{x}{a}-\frac{a}{x}$ by $\frac{1}{a}+\frac{1}{x}$.

(6) $\frac{x+y}{3}-\frac{y}{2}$ by $\frac{2x-y}{2a}$.

(8) $(\frac{1}{a}+\frac{1}{b})(\frac{1}{a}-\frac{1}{b})$ by $(\frac{1}{a^6}-\frac{1}{b^6})$.

(4) Simplification of Complex Fractions.

113. A complex fraction is a fraction which contains fractions in the numerator, or denominator, or both.

They are simplified as follows :—Reduce mixed quantities to the form of fractions; we shall then have one of the following forms :—

$$\frac{\frac{1}{a}}{b}, \quad \frac{\frac{c}{a}}{b}, \quad \frac{\frac{a}{b}}{c}, \quad \text{or} \quad \frac{\frac{a}{b}}{\frac{c}{d}},$$

which, by the rules for multiplication and division of fractions, become $\frac{b}{a}$, $\frac{bc}{a}$, $\frac{a}{bc}$, $\frac{ad}{bc}$.

If, however, the numerator and denominator be themselves complex fractions, these must be first simplified and we proceed step by step as in arithmetical fractions.

Examples.

$$\frac{\frac{1}{c+\frac{d}{y}}}{} = \frac{\frac{1}{\frac{cy+d}{y}}}{} = \frac{y}{cy+d} \qquad \frac{\frac{a}{c-\frac{d}{y}}}{} = \frac{\frac{a}{\frac{cy-d}{y}}}{} = \frac{ay}{cy-d}.$$

$$\frac{a+\frac{b}{x}}{c} = \frac{\frac{ax+b}{x}}{c} = \frac{ax+b}{cx}.$$

$$\frac{a+\frac{b}{x}}{c+\frac{d}{y}} = \frac{\frac{ax+b}{x}}{\frac{cy+d}{y}} = \frac{ax+b}{x}\cdot\frac{y}{cy+d} = \frac{(ax+b)y}{x(cy+d)}.$$

$$\frac{a+\dfrac{b}{x+\dfrac{c}{z}}}{c+\dfrac{d}{y+\dfrac{f}{g}}} = \frac{a+\dfrac{bz}{xz+c}}{c+\dfrac{dg}{gy+f}} = \frac{\dfrac{a(xz+c)+bz}{xz+c}}{\dfrac{c(gy+f)+dg}{gy+f}}$$

$$= \frac{(axz+ac+bz)(gy+f)}{(cgy+cf+dg)(xz+c)}.$$

Exercises. No. 35.

Simplify :—

(1) $\dfrac{1}{a-\dfrac{1}{b}}$.

(2) $\dfrac{1}{a-\dfrac{1}{b-\dfrac{1}{c}}}$.

(3) $\dfrac{x-\dfrac{1}{x}}{x-1-\dfrac{x-1}{1-\dfrac{x}{3+x}}}$.

(4) $\dfrac{z}{x-\dfrac{xy}{z+y}}$.

(5) $\dfrac{1-\dfrac{a^3}{x^3}}{\dfrac{1}{x^2}-\dfrac{a}{x^3}}$.

(6) $\dfrac{a+\dfrac{b-a}{1+ba}}{1-a\times\dfrac{b-a}{1+ba}}$.

(7) $\dfrac{a-c\times\dfrac{ae-bd}{ce-bf}}{b}$.

(8) $\dfrac{(a-\dfrac{b^2}{2a})(a-\dfrac{a^2+b^2}{a+b})}{1-\dfrac{a}{a+b}}$.

Prove that

$$\frac{(1-\frac{a}{b}+\frac{b}{a})(\frac{a+b}{2a}+\frac{a-b}{2b})}{(a-2b+\frac{b^2}{a})(\frac{a}{a+b}+\frac{b}{a-b})} = \frac{(ab+b^2-a^2)(a+b)}{2ab^2(a-b)}.$$

(5) Highest Common Factor and Lowest Common Multiple.

114. In Arithmetic we soon find that if in the addition of fractions we always follow the rule, to multiply the numerator and denominator of each fraction by all the denominators except its own, there is a waste of labour whenever two or more of the denominators

contain common factors. The denominator we want is the Least Common Multiple (L. C. M.) of the denominators, the number which just contains all their factors and no more. And in finding the L. C. M. we have our attention directed to the Greatest Common Measure (G. C. M.) of these numbers, viz. the greatest number which will divide them all. In Algebra we find the same necessity for economy of labour, and it has been usual to seek for the G. C. M., defined as the greatest exact divisor of two or more expressions, and the L. C. M., as the least expression which is exactly divisible by them. But the inappropriateness of these names to Algebra has been for some time recognized. In Algebra we are not dealing with the greatness of numerical magnitude, but with the higher or lower dimensions of expressions. Instead of G. C. M. the name Highest Common Factor (H. C. F.) has been employed and will be here adopted, and instead of *Least* we say *Lowest* Common Multiple (L. C. M.).

115. We will first consider simple algebraic expressions, such as $2a^2bx^3$, $21abx^3y$, $12ab^2xy^3$. The H. C. F. must contain every factor which is present in all of them, and no factor which is absent from any one of them. Hence this rule:—Write down every letter which is present in all, place over every letter the lowest index which it has in any one of them, and prefix the G. C. M. of the numerical coefficients. Thus the H. C. F. of the above expressions is abx.

116. Again, the lowest common multiple is divisible by every one of the expressions, and the quotients have no common factor. Hence the rule:—Write down every letter which occurs in any one of the expressions, place over each the highest index which occurs in any one of them, and prefix the arithmetical L. C. M. of the numerical coefficients. Thus the L. C. M. of the above expressions is $84a^2b^2x^3y^3$.

117. When two or more compound expressions can be resolved by inspection into binomial factors, the H. C. F. and the L. C. M. may be found by the above rules.

Thus the H. C. F. of $3(x-1)$, x^2-2x+1, $4x^3-4$ is $x-1$, and the L. C. M. is $3\times4(x-1)^2(x^2+x+1) = 12(x-1)(x^3-1)$.

Exercises. No. 36.

Find the H. C. F. and the L. C. M. of:—

(1) $6a^2b$ and $9ab^2$.
(2) $10x^3yz$ and $18xy^3z$.
(3) $32x^2y^3z^2$ and $48xy^4z$.
(4) $51a^2xyz^2$ and $111ax^3y^3z$.
(5) $3ab$, $5a^2b$, $15ab^3$.
(6) $21a^3x$, $49ax^3y$, $9xy^3$.
(7) $3pqxy^2$, $15qx^2y$, $81xy^3$.
(8) mx, ny, pz.
(9) mx^2yz, nxy^2z, $pxyz^2$.
(10) $28a^4x^3y^4z^5$, $64ax^4y^3z^8$, $91a^2x^2y^8z^2$, $120a^3x^8yz$.

Exercises. No. 37.

Find the H. C. F. and the L. C. M. of:—

(1) $2(x-a)$ and $3(x^2-a^2)$.
(2) $2(x+a)$, $8(x^2-2ax+x^2)$, $4(x^2-a^2)$.
(3) x^2-b^2, $3(x^2+2bx+b^2)$.
(4) x^3-a^3, x^2-a^2.
(5) $p(m^2-n^2)$, $q(m+n)$, $r(m-n)$.
(6) x^2+ax+a^2, $3(x^3-a^3)$.
(7) x^2+bx+b^2, $5(x-b)$, $6(x^3+b^3)$.
(8) $21(x^4-a^4)$, $14(x^2+a^2)$, $18(x^2-a^2)$.
(9) $x^2+9ax+14a^2$, x^2-4a^2.
(10) $x^2-6ax+9a^2$, $x^2+ax-12a^2$.

118. We now come to the H. C. F. of two compound expressions, which cannot readily be resolved into factors. Proceed as follows:—Arrange the expressions in descending powers of some one letter. Take one of them, in which the highest power is less or not greater than that in the other, and make it divisor, the other dividend. Take the remainder, if any, for a new divisor, the preceding divisor for dividend; and so on, continuing the operation until there is no remainder. The last divisor, freed from fractions and any common numerical factor, is the H. C. F.

In the course of the work any line may be multiplied by an

integer, to get rid of fractions, and we may divide out any factor, whether numerical or symbolical, which is common to every term. And when the first term of a remainder is negative, we may change the signs throughout before making it a divisor. If the two expressions have any common factor, which presents itself to the eye, it should be removed prior to the process, and restored afterwards.

119. The method depends on the following considerations:— If two expressions, which we may denote by A and B, contain the same factor, so will their sum or difference, or any expression of the forms $aA+bB$, or $aA-bB$, or, as we may write them together for shortness, $aA \pm bB$, where a and b are numerical or simple algebraical quantities. For, if $A = mC$, and $B = nC$,

$$aA \pm bB = amC \pm bnC = (am \pm bn).C,$$

which is made up of the algebraical factors C and $am \pm bn$. Therefore in the process of division, or in the process as modified, we can never get rid of the common factor; but we gradually lower the dimensions, still retaining it. Whenever first we arrive at a remainder of the dimensions of the factor, we can only have the factor itself, or the same multiplied by a numerical factor, and this will appear by its yielding a quotient without a remainder.

Having obtained this factor, we restore all factors which were removed at the beginning; if the coefficients are fractional we multiply the expression by a number which will make them integral, and if the first term be negative, when the expression is arranged on the same principle as the divisor and dividend, we change the signs of all the terms.

120. In order to define strictly the H. C. F., we shall lay down the following definitions:—Of all integral expressions, formed from $3x^2+5xy+2y^2$ by multiplying or dividing it by numerical or simple symbolical factors, we call the above, which contains no such factor, the primary integral expression; those formed by multiplying it, as $6ax^2+10axy+4ay^2$, multiple integral expressions; and those formed by dividing it, as $\frac{x^2}{2}+\frac{5xy}{6}+\frac{y^2}{3}$, sub-multiple

integral expressions. Now, by the H. C. F. we mean the highest multiple integral expression, that is a common factor of the divisor and dividend; and which, when arranged on the same principle as the divisor and dividend, has its first term positive.

121. The L. C. M. of two compound expressions may be found from the H. C. F.

Let D be the H. C. F. of two expressions A and B, and let $A = aD$, $B = bD$: then a and b have no common factor.

$$\therefore \text{ the L. C. M.} = abD = \frac{aD \cdot bD}{D} = \frac{AB}{D} = aB = bA,$$

which are all equivalent forms.

Thus the L. C. M. may be found as the product of the two expressions divided by the H. C. F., or, which is generally more convenient, by multiplying one expression by the quotient found on dividing the other by the H. C. F.

The method will be best understood by some examples.

Required the H. C. F. of

$$x^4-9x^3+7x^2+9x-8 \quad \text{and} \quad x^4+7x^3-9x^2-7x+8.$$

$$\begin{array}{r|l} x^4-9x^3+7x^2+9x-8 & x^4+7x^3-9x^2-7x+8 \;(\; 1 \\ & x^4-9x^3+7x^2+9x-8 \\ \hline & 16x^3-16x^2-16x+16 \end{array}$$

Removing the factor 16, we make this remainder a new divisor:

$$\begin{array}{r|l} x^3-x^2-x+1 & x^4-9x^3+7x^2+9x-8 \;(\; x-8 \\ & x^4-x^3-x^2+x \\ \hline & -8x^3+8x^2+8x-8 \\ & -8x^3+8x^2+8x-8 \\ \hline & * \quad * \quad * \quad * \\ \hline\hline \end{array}$$

x^3-x^2-x+1 is the required H. C. F., and the L. C. M. is $(x-8)(x^4+7x^3-9x^2-7x+8)$, which may be multiplied out.

Again, required the H. C. F. of

$$4x^4+9x^3+2x^2-2x-4 \quad \text{and} \quad 3x^3+5x^2-x+2.$$

We make the latter expression, which is of the lower dimensions, the divisor, and to avoid fractions we multiply the former by 3.

$$
\begin{array}{l}
\qquad\qquad 4x^4+\ 9x^3+\ 2x^2-\ 2x-\ 4 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 3 \\
\hline
3x^3+5x^2-x+2\ \big)\ 12x^4+27x^3+\ 6x^2-\ 6x-12\ \big(\ 4x \\
\qquad\qquad\ 12x^4+20x^3-\ 4x^2+\ 8x \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 7x^3+10x^2-14x-12 \\
\text{Again, multiplying by 3,} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 3 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 21x^3+30x^2-42x-36\ \big(\ 7 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 21x^3+35x^2-\ 7x+14 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad -\ 5x^2-35x-50
\end{array}
$$

We reject the factor -5,

$$
\begin{array}{l}
x^2+7x+10\ \big)\ 3x^3+\ 5x^2-\quad x+\ \ 2\ \big(\ 3x-16 \\
\qquad\qquad\ 3x^3+21x^2+\ 30x \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -16x^2-\ 31x+\ \ 2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -16x^2-112x-160 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 81x+162
\end{array}
$$

We reject the factor 81,

$$
\begin{array}{l}
x+2\ \big)\ x^2+7x+10\ \big(x+5 \\
\qquad\ x^2+2x \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 5x+10 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 5x+10 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad *\quad\ * \\
\hline\hline
\end{array}
$$

$x+2$ is the required H. C. F., and the L. C. M. is the product $(3x^2-x+1)(4x^4+9x^3+2x^2-2x-4)$.

The process may sometimes be shortened by various expedients. If the first coefficients are large and prime to each other at one end of the expression, we may sometimes gain by arranging them beginning at the other end, and it is often advantageous to lower the initial coefficient by employing an approximate quotient. The work above might have been continued as follows:

$$
\begin{array}{r|l}
3x^3+5x^2-x+2 & 12x^4+27x^3+6x^2-6x-12 \\
 & 12x^4+20x^3-4x^2+8x \\
\hline
 & 7x^3+10x^2-14x-12 \\
 & 6x^3+10x^2-2x+4 \\
\hline
 & x^3-12x-16
\end{array}
\qquad (4x+2
$$

$$
\begin{array}{r|l}
x^3-12x-16 & 3x^3+5x^2-x+2 \\
 & 3x^3-36x-48 \\
\hline
 & 5x^2+35x+50
\end{array}
\qquad (3
$$

and so continuing as before. No such siftings get rid of the important factor.

Exercises. No. 38.

Find the H. C. F. and L. C. M. of the following:—

(1) $2x^2-x-10$ and $3x^2+13x+14$.
(2) $21x^2+8x-45$ and $12x^2+5x-25$.
(3) $6a^2-ab-35b^2$ and $8a^2-2ab-45b^2$.
(4) $6x^2-18xy-60y^2$ and $2x^2-6xy-20y^2$.
(5) $9x^2+6xy-35y^2$ and $21x^2-41xy+10y^2$.
(6) $6x^3-21x^2-10x+35$ and $9x^3+15x^2-15x-25$.
(7) $2x^3+9x^2+16x+15$ and $3x^3+13x^2+23x+21$.
(8) $6x^3+5x^2-19x-20$ and $2x^3-15x^2+2x+35$.
(9) x^3+12x^2-121 and $3x^3+2x^2-34x+11$.
(10) $x^3+3x^2y-9xy^2+5y^3$ and $x^3-7x^2y+11xy^2-5y^3$.
(11) x^4-2x+1 and $x^4+2x^3+2x^2-1$.
(12) $6x^4-2x^3-3x^2+4x-1$ and $8x^4+2x^3-4x^2+3x+1$.
(13) $x^4+a^2x^2-6a^4$ and $2x^4+a^2x^2-10a^4$.
(14) $x^4+2x^3-7x^2+8x-5$ and $2x^4+5x^3-10x^2+14x-15$.
(15) $4x^4-x^2-2x-1$ and $6x^4-x^3-6x^2-4x+1$.
(16) $6x^4-x^3-3x^2-4x-4$ and $8x^4-2x^3-19x^2+3x+10$.
(17) $6x^5-9x^4+19x^3-12x^2+19x-15$ and
$4x^4-2x^3+10x^2+x+15$.
(18) $3x^5+2x^4+x^3$ and $3x^4+2x^3-3x^2+2x-1$.
(19) x^5-x^3-x+1 and $x^4-x^3+x^2-2x+1$.
(20) $2x^2-x^3-1$ and $x^3-x^4+2x^2-x-1$.

122. The following is an example of the method of finding the H. C. F. and L. C. M. of several expressions; viz. of

$$x^3+5x^2+2x-12,\ 2x^3-x^2-18x+20,$$
$$x^4+8x^3+17x^2-6x-36.$$

We first find the H. C. F. of

$$x^3+5x^2+2x-12, \text{ and } 2x^3-x^2-18x+20,$$

which is x^2+2x-4, and then of the H. C. F. so found and the remaining expression, which is still x^2+2x-4.

Again, the L. C. M. of

$$x^3+5x^2+2x-12 \text{ and } 2x^3-x^2-18x+20$$

is their product divided by

$$x^2+2x-4 \text{ or } (x+3)(2x-5)(x^2+2x-4),$$

and the L. C. M. of this expression and $x^4+8x^3+17x^2-6x-36$ is their product divided by

$$(x+3)(x^2+2x-4) \text{ or } (x+3)^2(2x-5)(x^2+2x-4).$$

Exercises. No. 39.

Find the H. C. F. and L. C. M. of

(1) $2x^2+x-3$, $3x^2-2x-1$, and x^3-x^2-x+1.

(2) $2x^2-x-3$, $2x^2-5x+3$, and $6x^2-13x+6$.

(3) $9x^2-9x-10$, $3x^2+x-10$, $3x^2-26x+35$.

(4) $21x^3+28x^2-15x-20$, $21x^3-28x^2-15x+20$, $14x^3-21x^2-10x+15$.

(5) x^2-9, x^2-6x+9, $x^3+3x^2-9x-27$.

(6) $x^2-3xy-10y^2$, $x^2+2xy-35y^2$, $x^2-8xy+15y^2$.

(6) Reduction of Fractions to their Lowest Terms.

123. Fractions are reduced to their lowest terms by dividing the numerator and denominator by all their common factors, or by their H. C. F.

Example. $\dfrac{14a^2-7ab}{10ac-5bc}=\dfrac{(2a-b)7a}{(2a-b)5c}=\dfrac{7a}{5c}$.

Example. $\dfrac{9x^3+53x^2-9x-18}{x^2+11x+30}=\dfrac{(x+6)(9x^2-x-3)}{(x+6)(x+5)}$

$$=\frac{9x^2-x-3}{x+5}.$$

Exercises. No. 40.

Reduce to their lowest terms the fractions:—

(1) $\frac{21a^3b^2c-9ab^3c^2}{15a^2b^2c-3a^5b^4c^2-12ab^2c}$.

(2) $\frac{12a^3x^4+2a^2x^5}{18ab^2x+3b^2x^2}$.

(3) $\frac{a^3-x^3}{(a-x)^2}$.

(4) $\frac{a^3+(1+a)ay+y^2}{a^4-y^2}$.

(5) $\frac{6ac+10bc+9ad+15bd}{6c^2+9cd-2c-3d}$.

(6) $\frac{x^2+2x-3}{x^2+5x-6}$.

(7) $\frac{9x^3+53x^2-9x-18}{x^2+11x+30}$.

(8) $\frac{2x^3+x^2-8x+5}{x^3+x^2-2x}$.

(9) $\frac{a^2+b^2+c^2+2ab+2ac+2bc}{a^2-b^2-c^2-2bc}$.

(10) $\frac{a^2-3ab+ac+2b^2-2bc}{a^2-b^2+2bc-c^2}$.

(11) $\frac{x^4-8x^3+6x^2+8x-7}{x^4+6x^3-8x^2-6x+7}$.

(12) $\frac{8x^3-10x^2-16x-3}{6x^4-22x^3+31x^2-23x-7}$.

124. Some further examples in the Addition and Subtraction of Fractions are now given. The method of procedure will in general be the following, as in Arithmetic:—Find, by whatever process may be the most convenient, the L. C. M. of all the denominators; this will be the common denominator of the resulting fraction. To find the corresponding numerators of the several fractions, divide the common denominator by the denominator of the first fraction, and multiply its numerator by the quotient, and so on for every successive fraction. The fractions are thus all reduced to a common denominator and their numerators may be added or subtracted, according to sign.

Example. $\frac{1}{a-x}+\frac{1}{a+x}$;

L. C. M. $=a^2-x^2$. Quotients $a+x$ and $a-x$;

$$\therefore \frac{1}{a-x}+\frac{1}{a+x}=\frac{a+x}{a^2-x^2}+\frac{a-x}{a^2-x^2}=\frac{2a}{a^2-x^2}.$$

Example. $\frac{1}{a^2-x^2}-\frac{1}{(a+x)^2}$;

L. C. M. $=(a-x)(a+x)^2$. Quotients $a+x$, $a-x$;

$$\therefore \frac{1}{a^2-x^2}-\frac{1}{(a+x)^2}=\frac{a+x-a+x}{(a-x)(a+x)^2}=\frac{2x}{(a-x)(a+x)^2}.$$

Sometimes it will be found convenient, where there are several fractions, to deal with two of them separately, and afterwards to bring in the others; for example, in (6) below.

Exercises. No. 41.

Add and simplify:—

(1) $\frac{x+y}{x-y}-\frac{x-y}{x+y}$.

(2) $\frac{1}{x^2-2x+1}+\frac{1}{x^2-3x+2}$.

(3) $\frac{1}{x^3-1}-\frac{1}{x^2+x+1}$.

(4) $\frac{3}{x^2+x+1}-\frac{3}{x^2-x+1}$.

(5) $\frac{x^2-x+1}{x^2+x+1}-\frac{x^3+1}{x^3-1}$.

(6) $\frac{1}{x-a}-\frac{1}{x+a}+\frac{3}{x^2-a^2}$.

(7) $\frac{1}{a-b+c}-\frac{1}{a+b-c}+\frac{1}{a-b-c}$.

(8) $\frac{x}{x-6}+\frac{x}{x-4}-\frac{2x}{x-8}$.

(9) $\frac{1}{x-2}-\frac{1}{x-3}+\frac{1}{x+1}$.

(10) $\frac{3a+2x}{a+x}-\frac{5a-x}{a-x}+\frac{a}{2x}$.

(11) $\frac{3}{1-2x}-\frac{7}{1+2x}-\frac{2-10x}{4x^2-1}$.

(12) $\frac{a^3}{(a+b)^3}-\frac{ab}{(a+b)^2}+\frac{b}{a+b}$.

(13) $\frac{x}{(x-1)^2}-\frac{1}{(x+1)^2}-\frac{x(x^2+3)}{(x^2-1)^2}$.

(14) $\frac{1}{(x-a)(x-2a)}-\frac{1}{(x-a)(x-3a)}+\frac{1}{(x-2a)(x-3a)}$.

(15) $\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-c)(b-a)}+\frac{c^2}{(c-a)(c-b)}$.

(16) $\frac{2ax+x^2}{(a-x)^2}-\frac{a^2+5ax}{(a+x)^2}-\frac{x}{(a-x)}$.

(17) $\frac{3}{4(1-x)^2}+\frac{3}{8(1-x)}+\frac{1}{8(1+x)}-\frac{1-x}{4(1+x^2)}$.

(18) $\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$.

(19) $\frac{\sqrt{1-x}+\frac{1}{\sqrt{1+x}}}{1+\frac{1}{\sqrt{1-x^2}}}$.

CHAPTER VI.

RATIO AND PROPORTION.

(1) General Notions of Ratio and Proportion.

125. As explained in Arithmetic, *the relative magnitude of two quantities, measured by the number of times (or parts of times) which the one contains the other, is called their ratio.* The ratio of a to b is expressed by the notation $a : b$, a being called the first term (or antecedent) of the ratio, b the second term (or consequent). But the ratio being measured by the fraction $\frac{a}{b}$, that fraction is constantly used in Algebra to denote the ratio.

126. The term Proportion is used in common life to express the true correspondence of parts in two corresponding objects under comparison, such as a picture and its pattern; or two maps constructed on different scales. If a corresponding pair of lines be selected out of each, the ratio of the first pair ought to be the same as the ratio of the second pair. *Hence proportion is defined to be the equality of ratios.* The four quantities which constitute a proportion are said to be proportionals.

127. A proportion between four quantities, a, b, c, d, is represented thus: $a : b :: c : d$ (by ratios)

or thus: $\frac{a}{b} = \frac{c}{d}$ (by fractions)

and is read thus: a is to b as c to d.

Both notations assert the equality of the two ratios, a to b, and c to d. In Algebra however it is to the fractional notation that we have to refer, in order to elicit the properties of proportions. If we translate the equality $\frac{a}{b} = \frac{c}{d}$ into words, it amounts to this, which may be taken as the definition of proportionals:—*Four quantities are proportionals, when the first is the same multiple, part, or parts of the second, which the third is of the fourth.*

(2) Properties of Proportionals.

128. Let a, b, c, d be four magnitudes in proportion; then

$$\frac{a}{b} = \frac{c}{d}. \quad \text{(i)}$$

We may multiply both sides of the equality by the same quantity bd, since if equals be multiplied by equals, the results are equal.

Hence $$\frac{a}{b} \times bd = \frac{c}{d} \times bd;$$

or $$ad = bc. \quad \text{(ii)}$$

Now a and d are the *extreme* (i.e. the outside) terms of the proportion, and b and c are the *mean* (i.e. the middle) terms of the proportion, whence we learn that—*If four quantities are in proportion, the products of the extremes and the means are equal.* And conversely,—*If four quantities placed in order are such that the products of the extremes and means are equal, they are proportionals.*

The property of the extremes and means (ii) often furnishes in Algebra, as in Arithmetic, a more convenient test of the proportionality of four quantities than (i).

129. We may now use the property of the extremes and means to find a fourth proportional to three quantities a, b, c; that is, to find a quantity x, such that $a : b :: c : x$.

By (ii), $$ax = bc, \qquad \therefore \ x = \frac{bc}{a}.$$

Hence we simply multiply together the second and third, which represent the means, and divide by the first. (Rule of Three.)

If $a : b :: b : c$, a, b, c are said to be in *continued proportion*, and $b^2 = ac$. And b is also said to be a *mean proportional* to a and c; and, again, c is said to be a *third proportional* to a and b.

130. We may deduce from the simple properties of fractions various arrangements of the quantities a, b, c, d, which will at once present us with other proportions. Of these the most useful are the following, some of which are often quoted by the annexed phrases:—

(1) Since $\frac{a}{b} = \frac{c}{d}$, $\therefore \frac{a}{b} \times \frac{b}{c} = \frac{c}{d} \times \frac{b}{c}$, or $\frac{a}{c} = \frac{b}{d}$: that is,

$$a : c :: b : d. \qquad \text{(Alternando.)}$$

(2) Since $\frac{a}{b}=\frac{c}{d}$, $\therefore\ 1\div\frac{a}{b}=1\div\frac{c}{d}$, or $\frac{b}{a}=\frac{d}{c}$: that is,

$$b : a :: d : c. \qquad \text{(Invertendo.)}$$

(3) Since $\frac{a}{b}=\frac{c}{d}$, $\therefore\ \frac{a}{b}+1=\frac{c}{d}+1$, or $\frac{a+b}{b}=\frac{c+d}{d}$: that is,

$$a+b : b :: c+d : d. \qquad \text{(Componendo.)}$$

(4) Since $\frac{a}{b}=\frac{c}{d}$, $\therefore\ \frac{a}{b}-1=\frac{c}{d}-1$, or $\frac{a-b}{b}=\frac{c-d}{d}$: that is,

$$a-b : b :: c-d : d. \qquad \text{(Dividendo.)}$$

(5) Since $\frac{a}{b}=\frac{c}{d}$ and $\frac{a-b}{b}=\frac{c-d}{d}$, $\therefore\ \frac{a}{a-b}=\frac{c}{c-d}$: that is,

$$a : a-b :: c : c-d. \qquad \text{(Convertendo.)}$$

(6) Since $\frac{a+b}{b}=\frac{c+d}{d}$ and $\frac{a-b}{b}=\frac{c-d}{d}$, $\therefore\ \frac{a+b}{a-b}=\frac{c+d}{c-d}$;

that is, $a+b : a-b :: c+d : c-d$. (Componendo and Dividendo.)

(7) Since $\frac{a}{b}=\frac{c}{d}$, we have $\frac{ma}{mb}=\frac{nc}{nd}$; whence we obtain

$$ma : mb :: nc : nd,$$

where m and n are general symbols.

(8) Since $\frac{a}{b}=\frac{c}{d}$, we have $\frac{ma}{nb}=\frac{mc}{nd}$; from which we obtain

$$ma : nb :: mc : nd.$$

(9) Since $\frac{a}{b}=\frac{c}{d}$, we have $\left(\frac{a}{b}\right)^m=\left(\frac{c}{d}\right)^m$, or $\frac{a^m}{b^m}=\frac{c^m}{d^m}$; whence

$$a^m : b^m :: c^m : d^m,$$ m being a general symbol.

(10) If $\frac{a}{b}=\frac{c}{d}$, and $\frac{c}{d}=\frac{e}{f}$, then $\frac{a}{b}=\frac{e}{f}$; or if $a : b :: c : d$, and

$$c : d :: e : f; \quad \text{then} \quad a : b :: e : f.$$

(11) Let $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\ldots=k$; then

$$\begin{aligned} a &= bk, \quad \text{and} \quad ma = mbk, \\ c &= dk, \qquad\quad nc = ndk, \\ e &= fk, \qquad\quad pe = pfk; \end{aligned}$$

$$\therefore\ a+c+e+\ldots = (b+d+f+\ldots)k,$$

and $\quad ma+nc+pe+\ldots = (mb+nd+pf+\ldots)k;$

$$\therefore\ \frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\ldots=\frac{a+c+e+\ldots}{b+d+f+\ldots}=\frac{ma+nc+pe+\ldots}{mb+nd+pf+\ldots},$$

which may be read also as a proportion.

This method will often be found useful.

(12) If $a : b :: b : c$; then $\frac{a}{b}=\frac{b}{c}$; and

$$\frac{a}{c}=\frac{a}{b}\times\frac{b}{c}=\frac{a}{b}\times\frac{a}{b}=\frac{a^2}{b^2};\quad \text{i.e. } a : c :: a^2 : b^2;$$

or the first has to the third the same ratio that the square of the first has to the square of the second.

(13) Again, if four magnitudes a, b, c, d be in continued proportion, so that $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$; then will

$$\frac{a}{d}=\frac{a}{b}\times\frac{b}{c}\times\frac{c}{d}=\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}=\frac{a^3}{b^3};$$

or the first has to the fourth the same ratio that the cube of the first has to the cube of the second.

The two results here obtained answer to the Geometrical definitions of *Duplicate* and *Triplicate* ratio.

Many other like properties may be proved in a similar way.

Exercises. No. 42.

(1) If $a : b :: 3 : 2$, determine the ratio $a+b : a-b$.

(2) If $a : 6 :: b : 5$, find the ratio $a+b : a-b$.

(3) If $a : b :: 5 : 3$, find the value of $\frac{a}{b}+\frac{b}{a}$.

(4) If $a+b : a-b :: c : d$, prove that $(a+b)(c+d) = 2ac$.

(5) If $ad = bc$, then $a-b : c-d :: a : c$, and
$$ma+nc : mb+nd :: c : d.$$

(6) If $a : b :: c : d$, prove that $(\frac{a}{b})^2+(\frac{c}{d})^2 = 2\frac{ac}{bd}$.

(7) If $a : b :: c : d$ and a be $> b$, then $c > d$.

(8) If $a : b :: c : d$, and a is the greatest, d is the least.

(9) $a : b :: c : d$, and $e : f :: g : h$, prove that
$ae : bf :: cg : dh$. (Compounding the ratios.)

(10) If $a : b :: a' : b'$, and $b : c :: b' : c'$, prove that
$a : c :: a' : c'$. (Ex æquali).

131. Euclid's test of proportionality is as follows :—'Four quantities taken in order are proportionals, if there being taken any equimultiples whatever of the first and third, and any equimultiples whatever of the second and fourth—the multiple of the third is greater than, equal to, or less than the multiple of the fourth according as the multiple of the first is greater than, equal to, or less than the multiple of the second.'

The quantities here meant are concrete magnitudes, visible lengths, &c., not numbers. Let us suppose, however, subject to later remarks, that, being referred to some unit, they can be expressed as numbers, under the general symbols, a, b, c, d.

In algebraical language, this test is as follows :—a, b, c, d are proportionals if, any equimultiples whatever of a, c being taken (say ma, mc) and any equimultiples of b, d (say nb, nd), mc is $>$, $=$, or $<$ nd, according as ma is $>$, $=$, or $<$ nb.

The simplest case is that of equality. When two integers, m, n, can be found such that $ma = nb$, and $mc = nd$, the test makes a, b, c, d proportionals. They are so also by the algebraical test, since $\frac{a}{b} = \frac{n}{m} = \frac{c}{d}$. In this case a, b are proportional to integers, n, m: so are c, d. They may therefore properly be represented by so many units, or by algebraical symbols, as supposed above. Such quantities are said to be *commensurable.*

132. Since $\frac{a}{b} = \frac{c}{d}$ gives $\frac{ma}{nb} = \frac{mc}{nd}$, it follows that mc will be $>$, $=$, or $<$ nd, according as ma is $>$, $=$, or $<$ nb. Thus Euclid's test of proportion follows from the algebraical test.

133. Conversely, the algebraical test follows from Euclid's test. For, suppose that a, b, c, d are proportionals according to Euclid's test, then shall $\frac{a}{b} = \frac{c}{d}$.

For, if not, one of these fractions must be greater than the other.

Let $\frac{a}{b} > \frac{c}{d}$; and since it must be possible to find some fraction between them, let $\frac{n}{m}$ be such a fraction. Then $\frac{a}{b} > \frac{n}{m}$ and $\frac{n}{m} > \frac{c}{d}$.

$$\therefore\ ma > nb \text{ and } mc < nd.$$

But this is contrary to Euclid's test, which is against the hypothesis; therefore $\frac{a}{b}$ cannot be unequal to $\frac{c}{d}$, or $\frac{a}{b} = \frac{c}{d}$.

(3) Incommensurables.

134. Geometry shows us certain magnitudes, for example, the side and diagonal of a square, which are not in the ratio of any two integers, and the ratio of which can never be completely expressed by any finite number of figures. Such quantities are said to be *incommensurable*.

Euclid's test was devised to meet the difficulties arising from incommensurable quantities.

The test supposes not one trial, for any one set of values of m and n, taken at random; but an infinite number, for all possible values of m and n, and any one failure of the test would show that a, b, c, d are not proportionals. Such an infinite number of trials being of course impossible, Euclid only employs the test where it can be shown from the properties of the magnitudes under consideration that the test is in all cases satisfied. It is sometimes supposed that the difficulties arising from incommensurable quantities, which Euclid has ingeniously evaded by the above test, do not arise when the algebraical definition is employed. This is however a mistake. In the first place, the reasoning which justifies the use of general symbols, as a, b, c, d, to represent incommensurables, in other words, interminable non-recurring decimals, is by no means simple. It can only be incompletely indicated here. Many operations, such as finding a mean proportional between two quantities (129), must frequently present us with such quantities, where we know that the value sought is perfectly definite. The result thus found (and other like results) can be made, by taking figures enough, to have an error less than any, however small, that we choose to assign. The inference is, the

method is true. And it is the principle of Algebra to admit all results which it can interpret, and to extend the meaning of its symbols accordingly. Again, were we to attempt to prove a, b, c, d proportionals by examining the equality of the products ad, bc, when a and b are incommensurable and likewise c and d, we could only obtain some such results as follow:—

$ad = \cdot 3156\ldots$ to 4 places, $\quad bc = \cdot 3156\ldots$ to 4 places,

the results, if we carried them further, still keeping parallel to each other, but never terminating, and never giving an absolute demonstration of equal ratios, since a different figure in the two products after any number of steps would suffice to prove that the quantities were not strictly proportionals. All that can be said then is, that the test (so far as we depend upon it) shows the four quantities to be approximately proportionals. But, of course, the very form of expressions will frequently prove that they are proportionals.

(4) Variation.

135. The word proportional is also applied in a somewhat different sense to indefinite quantities, such as Length, Width, Cost, Quantity; in such general maxims as—'The Cost is proportional to the Quantity,' 'The Length ought to be in proportion to the Breadth;' and it has been pointed out in Arithmetic, *Scholar's Arithmetic*, XII. (2), that from such maxims are derived the proportional quantities which we make use of in calculations. Thus, for instance, if we have two 'Quantities of material' and two corresponding 'Costs,' one of which is at present unknown, we find its value by Rule of Three, on the assumption that the four are proportionals, and so on, in all other cases. Now, instead of the above use of the word proportional—the usual phrase in Algebra and the Higher Mathematics is, The Length varies as the Breadth—the Cost varies as the Quantity. One quantity is said to vary as another *directly*, if it not only so depends upon it that the one cannot change without the other, but the two change together in such a manner that if one is doubled, trebled, or raised to any other multiple, the second is also doubled, trebled, or raised to the

corresponding multiple. But the one is said to vary as the other *inversely*, if when the one is doubled, or multiplied by any number, the other is halved or divided by the aforesaid number.

136. The notation for 'varies as' is $\propto$, so that $x \propto y$ is read x varies as y. Since the conditions expressed by x and y are satisfied when $x = cy$, where c is a constant quantity, we may always replace the relation between x and y by such an equality (and conversely). If any pair of corresponding values of x and y are known, c can be determined. For instance, let $y = 2$ when $x = 6$,

$$\therefore\ 6 = c \times 2, \quad \text{and} \quad c = 3; \qquad \therefore\ x = 3y;$$

and x becomes $= 9$, when $y = 3$.

137. The following are the most important propositions :—

(i) If $A \propto B$, and $B \propto C$, then will $A \propto C$.
For $A = mB$ and $B = nC$, m and n being constants.
$\therefore\ A = mnC$ or $A \propto C$, since mn is a constant.

(ii) If $A \propto C$, and $B \propto C$: then will $A \pm B \propto C$, and $\sqrt{AB} \propto C$.
For, if $A = mC$, and $B = nC$, then
$A \pm B = (m \pm n)C$ or $A \pm B \propto C$.
Also $AB = mnC^2$, and $\therefore\ \sqrt{AB} = \sqrt{mn}.C$;
or $\sqrt{AB} \propto C$.

(iii) If $A \propto B$ when C is invariable, and $A \propto C$ when B is invariable, then will $A \propto BC$, when both B and C are variable.

Let $A = A_1$, when $B = B_1$, and $C = C_1$, and become A_2 when B becomes B_2, C remaining C_1, and become A_3 when B remaining B_2, C becomes C_2.

Then $A_1 : A_2 :: B_1 : B_2$ or $\dfrac{A_1}{A_2} = \dfrac{B_1}{B_2}$.

Also $A_2 : A_3 :: C_1 : C_2$ or $\dfrac{A_2}{A_3} = \dfrac{C_1}{C_2}$.

$$\therefore\ \frac{A_1}{A_3} = \frac{B_1 C_1}{B_2 C_2}.$$

$\therefore$ A_1 changes to A_3 when B_1C_1 are together changed into B_2C_2, and $A_1 : A_3 :: B_1C_1 : B_2C_2$.

$$\therefore A \propto BC.$$

Similarly, if $A \propto B$, when C and D are unchanged,

$\propto C$, when B and D are unchanged,

$\propto D$, when B and C are unchanged,

then $A \propto BCD$ when B, C, D all vary.

For instance, the weight of a right solid varies as the length, and the breadth, and the height, separately; therefore the weight $\propto$ length $\times$ breadth $\times$ height, jointly, when all change.

Supposing A_3 to be unknown, and to be sought for, A_1, B_1, C_1, B_2, C_2 being given, we have the ordinary problem of Compound Proportion in Arithmetic, solved under the form

$$B_1C_1 : B_2C_2 :: A_1 : \text{Answer};$$

$$\text{or} \quad \text{Answer} = A_1 \times \frac{B_2}{B_1} \times \frac{C_2}{C_1};$$

with additional ratio-multipliers, if there are more variable elements. If A varies inversely as one of these elements, say D, or $A \propto \frac{1}{D}$, the corresponding ratio-multiplier will be inverted, as $\frac{D_1}{D_2}$.

CHAPTER VII.

RECAPITULATORY QUESTIONS AND EXERCISES.

138. The scholar should frequently run his eye over the various headings in the previous chapters, and should question himself whether he remembers the results and rules which have been established under each heading. He should also frequently run through the Exercises, especially the later ones, from Article 81, endeavouring to catch the results, except in the more lengthy examples, by inspection, as in Mental Arithmetic.

Questions and Exercises. No. 43.

(1) What is the Rule of Signs? To which of the simple processes, Addition, Subtraction, Multiplication, and Division does it apply? Give examples in each case.

(2) Give the Rules of Indices for Multiplication, and for Division, with examples.

Remark.—Since an algebraical symbol may be positive or negative, we may change a symbol, as $+a$ into $-a$, in any equality, if we do so throughout the equality.

(3) What do the formulae,

$$(a+b)^2 = a^2+2ab+b^2, \quad (a+b)^3 = a^3+3a^2b+3ab^2+b^3,$$

become when $+b$ is changed into $-b$?

(4) From the formula,

$$(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca,$$

obtain the formula for $(a+b-c)^2$, $(a-b+c)^2$, $(a-b-c)^2$.

Remark.—Since an algebraical symbol may have any value, we may write for b, $b+c$, or any other value, if we do so consistently throughout an equality.

(5) From the formula for $(a+b)^2$ find that for $(a+b+c)^2$.

(6) In the same way find that for $(a+b+c+d)^2$.

(7) Find the sum of $(a+b+c)^2$, $(a+b-c)^2$, $(a-b-c)^2$.

(8) Prove that

$$(a-b)^3+(b-c)^3+(c-a)^3 = 3(a-b)(b-c)(c-a).$$

(9) $(a+b+c)^3+(a+b-c)^3-2(a+b)^3 = 6(a+b)c^2$.

(10) $a^4+b^4+c^4+2a^2b^2+2b^2c^2+2c^2a^2 = (a^2+b^2+c^2)^2$.

(11) $a^4+b^4+c^4+2a^2b^2-2b^2c^2-2c^2a^2 = (a^2+b^2-c^2)^2$.

(12) $a^4+b^4+c^4-2a^2b^2-2b^2c^2-2c^2a^2 =$

$$(a+b+c)(a+b-c)(a-b+c)(a-b-c).$$

(13) $(a^2+c^2-b^2)^2+(a^2-c^2)^2-(a^2-c^2)^2 =$

$$(a+b+c)(a+b-c)(a-b+c)(a-b-c).$$

(14) $(x-y)^4+(y-z)^4+(z-x)^4 = 2(x^2+y^2+z^2-yz-zx-xy)^2$.

(15) Find the values, when $x = 3$, of

$$x^3+8x^2-5x+9, \quad x^4-17x^2+5x-11.$$

(16) If $$\frac{a}{b}=\frac{c}{d}=\frac{e}{f},$$

prove, as in Art. 130, (11), that each of these fractions

$$=\frac{a+c+e}{b+d+f}=\sqrt[3]{\frac{ace}{bdf}}=\sqrt[4]{\frac{a^4+c^4+e^4}{b^4+d^4+f^4}}.$$

(17) Prove, in a similar manner to the above, that if $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}$ be unequal fractions arranged in descending order of magnitude,

$$\frac{a+c+e+g}{b+d+f+h} > \frac{g}{h} \text{ and } < \frac{a}{b}.$$

(18) What sort of a magnitude is a ratio? and how is its value measured? Shew which is the greater ratio, 3 : 7 or 7 : 15.

(19) Prove that, c being positive, $\frac{a+c}{b+c} >$ or $< \frac{a}{b}$, according as $a <$ or $> b$.

(20) Definition. The ratio $a : b$ is said to be a ratio of greater or less inequality according as $a >$ or $< b$.

Prove that a ratio of greater inequality is diminished, and a ratio of less inequality increased, by adding any positive quantity to both terms of the ratio.

(21) Prove that $a^2 - 2ab + b^2 > 0$, and that the sum of any quantity and its reciprocal is greater than 2.

(22) If a, b, c, d be in continued proportion, prove that

$$\frac{(ac-bd)(ab-cd)}{ad+bc}=\frac{(a+d)^2-(b+c)^2}{2}.$$

(23) If x varies as y, prove that x^2+y^2 will vary as x^2-y^2.

(24) If $x \propto y$, and $x = 2$ when $y = 3$, find the relation between x and y.

(25) A varies partly directly as B and partly inversely as B. If A is 3 when B is 1, and also when B is 2, what will be the value of B when A is $4\frac{1}{2}$?

(26) Illustrate 'joint variation,' Art. 137, (iii), in solving the following question in Compound Proportion:—If a man travels 45 miles in 3 days by walking 4 hours a day, in what time will he travels 560 miles by walking 7 hours a day

(27) From a cask of wine the m^{th} part is drawn off and the cask filled up with water. If the same operation be repeated n times, find what fraction of the original quantity of wine will be left in the cask?

Exercises. No. 44.

Translate into words, as in Articles 81, 82, 83, the following equalities :—

(1) $\dfrac{x^3-y^3}{x-y} = x^2+xy+y^2$.

(2) $x^4+4 = (x^2+2x+2)(x^2-2x+2)$.

(3) $x^2+64 = (x^2+4x+8)(x^2-4x+8)$.

(4) $x^4+4y^4 = (x^2+2xy+2y^2)(x^2-2xy+2y^2)$.

(5) $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$.

(6) $(a^2+c^2)(b^2+d^2) = (ab+cd)^2+(ad-bc)^2$.

(7) Translate into words all the proportions (1) to (13) in Article 130.

Exercises. No. 45.

Translate into algebraic language the following :—

(1) The difference of the eighth powers of two numbers divided by the difference of the numbers is equal to the product of the sum of the numbers into the sum of their squares into the sum of their fourth powers.

(2) If four numbers are proportionals, the sum of the first and second is to their difference as the sum of the third and fourth to their difference.

(3) A quantity consisting of the sum of the squares of three numbers and of twice the product of every two of them is equal to the square of their sum.

(4) The sum of three numbers multiplied by the excess of every two of them above the third is equal to twice the sum of the products of the squares of every two of them less the sum of their fourth powers.

PART II.

APPLICATIONS OF ALGEBRA.

139. THE design of Part II of this work is to illustrate, by means of a few rudimentary applications of Algebra, the nature of the advantages derived from the use of general symbols. The same subject will be more fully developed in Part III. In both these Parts the order of arrangement will be found to be that sketched out in (10), which should now be referred to.

The general symbols will in all cases be subject to the laws of combination investigated in Part I: but occasion will be taken from time to time to explain some extensions of Algebraic notation.

CHAPTER VIII.

ON ALGEBRAICAL FORMULAE.

140. One of the most obvious advantages of Algebra is that it enables us briefly to record a whole class of results in a general formula. A Formula is an algebraical expression, consisting of general symbols, by means of which the solution of some question proposed is given in terms of its data. Such a formula embraces all particular cases, to any one of which it is easily adapted, by assigning to the symbols their appropriate particular values.

We commence by showing the manner in which Algebra enables us to express some of the rules and results of Arithmetic.

(1) Simple Interest.

141. The common rule in Arithmetic for finding the Interest on money lent is:—Multiply the principal (or sum lent) by the rate per cent. and divide by 100. The result will be the Interest for one year. Multiply the above result by the number of years for which Interest is required.

The rate per cent. or fixed sum to be paid for £100 lent is commonly used in business: but we shall obtain a simpler formula by taking £1 instead of £100 as the unit of Principal on which to compute interest.

Let £r (r being a fraction) be the number of £'s paid as interest on £1 for 1 year. (Note that r, the rate per £ = rate per cent. ÷ 100.) Then, if £P be the principal, and £I the interest for n years, £M the amount, or principal + interest, we have—

Interest for one year $= Pr$;

Interest for two years $= 2Pr$, and generally,

Interest for n years, or $I = Pnr$; (i)

$$M = P + I = P(1 + nr). \quad \text{(ii)}$$

142. The symbols represent positive quantities, integral or fractional. Observe that nothing is here proved beyond what was already proved in Arithmetic, but it is expressed in a short, simple form, easily remembered, and easily applied.

Suppose $P = 200$, $n = 2$, $r = \frac{1}{20}$ (equivalent to 5%).

Then $I = 200 \times 2 \times \frac{1}{20} = 20$, $M = 220$.

143. Any three of the 5 quantities P, I, M, n, r being given (except P, M, I which are not independent) the rest may be found. This includes 9 cases.

Thus, given $M = 220$, $n = 2$, $r = \frac{1}{20}$; then

$$220 = P(1 + \tfrac{2}{20});$$

$$220 = P \times \tfrac{22}{20};$$

$$10 = P \times \tfrac{1}{20};$$

$$P = 200, \quad I = 20.$$

Exercises. No. 46.

Apply the algebraical formulae in the following cases:—

(1) What is the interest for 1 year at 5 % on £210 10*s.*?

(2) At what rate per cent. will £230 8*s.* 4*d.* amount to £276 10*s.* in 4 years' time?

(3) Find the interest on £621 13*s.* for 29 days at 4 %.

(4) Find what principal will amount to £532 in 4 years at 3 %.

(5) In what time will £71 5*s.* amount to £88 7*s.* at 3 %?

(6) Give the values of the 5 quantities M, P, I, n, r involved in the solution of the last question, and state the nine questions which might be proposed, three of these quantities being known, to find the fourth.

(2) Discount.

144. When a sum of money is advanced upon a security not yet payable, the abatement made from the Gross Amount is called the Discount. The nett payment is called the Present Worth. Discount, reckoned according to strict theory, is such that if the Present Worth were put out to Interest, the Present Worth and Discount would together make the Gross Amount. Therefore the Discount = Interest upon the Present Worth.

Let P = Present Worth (or Nett Amount);

M = Gross Amount;

D = Discount for n years;

r = Rate per £., all in £'s.

$\therefore D = Pnr.$

$$P + D = P + Pnr = M;$$

$$P = \frac{M}{1 + nr};$$

$$D = \frac{Mnr}{1 + nr}.$$

145. In trade, Discount is commonly calculated as Interest on the Gross Amount, and this is necessarily greater than Interest on the Nett.

In fact, $I-D = Mnr - \frac{Mnr}{1+nr} = \frac{Mn^2r^2}{1+nr}$,

$= Dnr =$ Interest on the Discount,

or $= \frac{Inr}{1+nr} =$ Discount on the Interest.

Example.—What is the Present Worth of £200 due 2 years hence, reckoning interest at 5 %?

Here $M = 200, \quad n = 2, \quad r = \frac{5}{100} = \frac{1}{20}$.

$$P = \frac{200}{1+2.\frac{1}{20}} = \frac{200}{\frac{11}{10}} = \frac{2000}{11},$$

$$= £181\ 16s.\ 4\tfrac{4}{11}d.$$

$$D = M-P = £18\ 3s.\ 7\tfrac{7}{11}d.$$

Exercises. No. 47.

Apply the algebraical formulae to find:—

(1) Discount on £375 for 1 year at 5 %.

(2) Discount on £732 for 6 months at 4 %.

(3) Present Worth of £701 due 3 months hence at 5 %.

(4) Present Worth of £72 due after 141 days at 4 %.

(3) Compound Interest.

146. Compound Interest is when the Interest is added to the Principal after the lapse of a certain period (as one year) and so becomes part of the sum on which Interest is payable.

Let M be the amount at Compound Interest after n years.

Here, at the end of 1 year the amount $= P(1+r)$.

Let R denote the amount of £1 after 1 year $= 1+r$.

Then, at the end of 1 year, the amount $= PR$,

„ „ 2 years, „ $= PR \times R = PR^2$,

„ „ 3 years, „ $= PR^2 \times R = PR^3$,

„ „ n years, „ $= M = PR^n$.

If I be the Compound Interest at the end of n years,

$$I = M-P = P(R^n - 1).$$

Example.—Find the amount of £500 after 3 years at 4 % compound interest.

Here $P = 500$, $r = \frac{4}{100} = .04$, $R = 1.04$;

$\therefore M = 500 \times (1.04)^3$;

$= £562.432 = £562\ 8s.\ 7\frac{1}{2}\frac{7}{8}d.$

Exercises. No. 48.

(1) Find the compound interest on £375 at 3 % for 3 years.

(2) Find the compound interest on £407 16*s.* for 4 years at 4 %.

(3) Find the amount of £100 after 4 years at $3\frac{1}{2}$ % compound interest.

(4) Formulae of Mensuration.

147. The following simple results of Geometry, often studied under the name of Mensuration, are examples of the employment of the general symbols of Algebra.

Any linear unit having been chosen, the superficial and solid units will be respectively the square and the cube, which have their sides equal to the linear unit. See note, p. 13.

(1) Area of a square (side a) $= a^2$.

(2) Area of a rectangle (length a, breadth b) $= ab$.

(3) Area of a triangle (base a, altitude h) $= \frac{1}{2}ha$.

(4) Circumference of a circle (radius r) $= 2\pi r$, where π stands for an interminable decimal, approximately 3·14159, or more roughly $= 3\frac{1}{7}$.

(5) Area of circle (radius r) $= \pi r^2$.

(6) Volume of a cube (side a) $= a^3$.

(7) Volume of a right solid (length a, breadth b, height c) $= abc$.

(8) Surface of a sphere (radius r) $= 4\pi r^2$.

(9) Volume of a sphere (radius r) $= \frac{4}{3}\pi r^3$.

(10) Volume of a cone (radius of base r, height h) $= \frac{1}{3}\pi r^2 h$.

148. In any particular case of a square, circle, &c. we have only to substitute in the formula the appropriate value of a, r, &c., and to work out the result by ordinary Arithmetic.

Example.—Required the surface of a sphere, 1 foot in diameter.

Here $r = \frac{1}{2}$, the linear unit being 1 foot,

Surface $= 4\pi \times \frac{1}{4} = \pi = 3.14159$ square feet.

(5) Arithmetic Progression.

149. A series of quantities is said to be in Arithmetic Progression (in A. P.) when between every successive pair of terms there is a fixed or common difference.

Thus, $1, 3, 5, 7, \ldots$

$-2, 4, 10, 16, \ldots$

$a, a+b, a+2b, a+3b, \ldots$

$a, a-b, a-2b, a-3b, \ldots$

are all arithmetical progressions, the common difference in the first being 2, in the second 6, in the third b, in the fourth $-b$.

150. To find the n^{th} term and the sum of n terms of an Arithmetic Progression.

Let a denote the first term, d the common difference, l the n^{th} term, and s the sum of n terms: then the series will be

$$a, \quad a+d, \quad a+2d, \quad a+3d, \ldots$$

the coefficients of d counting in succession 1, 2, 3, &c., but being always 1 behind the number of the term; hence the n^{th} term

$$l = a+\overline{n-1}\,d. \qquad \text{(i)}$$

Also, $s = a+(a+d)+(a+2d)+ \ldots +(a+\overline{n-1}\,d)$,

and, reversing the order of the terms, we have also

$$s = (a+\overline{n-1}\,d)+(a+\overline{n-2}\,d)+ \ldots +(a+d)+a,$$

whence, adding together the corresponding terms, we find

$$2s = (2a+\overline{n-1}\,d)+(2a+\overline{n-1}\,d)+ \ldots +(2a+\overline{n-1}\,d),$$

$= n(2a+\overline{n-1}\,d)$, since there are n equal terms;

$$\therefore \quad s = \tfrac{1}{2}n(2a+\overline{n-1}\,d). \qquad \text{(ii)}$$

Otherwise, $s = \frac{1}{2}n(a+l)$. (iii)

Example.—Required the sum of 5, 8, 11, ... to 8 terms.

We find on trial that there is a fixed difference 3. Also, $a=5$, $n=8$;

$\therefore \quad s = 4(10+7\times 3) = 124$, by (ii); and $l = 26$, by (i).

Exercises. No. 49.

Find the last term and the sum of the following series:—

(1) 1, 3, 5, ... to 7 terms. (3) -1, 1, 3, ... to 6 terms.

(2) 2, 5, 8, ... to 11 terms. (4) 16, 13, 10, ... to 7 terms.

(5) 14, 8, 2, ... to 5 terms. Give all the terms.

(6) $a+3b$, $a+5b$, $a+7b$, ... to 6 terms.

In the following cases find l and s:—

(7) $a = 1$, $d = 1$, $n = 14$. (11) $a = \frac{3}{4}$, $d = \frac{1}{8}$, $n = 26$.

(8) $a = 2$, $d = 3$, $n = 17$. (12) $a = -7$, $d = 3$, $n = 8$.

(9) $a = 7$, $d = \frac{1}{4}$, $n = 16$. (13) $a = \frac{1}{2}$, $d = -\frac{1}{8}$, $n = 20$.

(10) $a = 2\frac{1}{2}$, $d = \frac{1}{3}$, $n = 100$. (14) $a = 0$, $d = \frac{1}{2}$, $n = 11$.

(15) $a = -10$, $d = -2$, $n = 6$. Give all the terms.

(6) Geometric Progression.

151. A series of quantities is said to be in Geometric Progression (in G. P.) when any two successive terms have a fixed or common ratio.

Thus, $1, 3, 9, 27, \ldots$

$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

$1, -2, 4, -8, \ldots$

$a, ab, ab^2, ab^3, \ldots$

are in G. P., the common ratios being $3, \frac{1}{2}, -2, b$.

152. To find the n^{th} term and the sum of n terms of a G. P.

Let a denote the first term of a G. P., r the common ratio, l the n^{th} term, and s the sum of n terms: then the series will be

$$a, \quad ar, \quad ar^2, \quad ar^3, \ldots$$

and the index of r in each term is one less than the number of the term in the series $\qquad l = ar^{n-1}. \qquad (i)$

Again, $\qquad s = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}.$

Also, $\qquad rs = \quad ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n.$

By subtraction, $(r-1)s = a(r^n - 1)$, if r be > 1;

$$\therefore \; s = a\frac{r^n - 1}{r - 1}. \qquad (ii)$$

Or, $\qquad (1-r)s = a(1-r^n)$, if r be < 1;

$$\therefore \; s = a\frac{1 - r^n}{1 - r}. \qquad (iii)$$

Example.—To sum 2, 6, 18, ... to 4 terms.

We find on trial that there is a fixed ratio 3. Also, $a = 2$, $n = 4$;

$\therefore s = 2 \times \frac{3^4 - 1}{3 - 1} = 80$, by (ii). Also, $l = 2 \times 3^3 = 54$, by (i).

153. In case (iii), if n be very great, r^n is very small, and n may be taken large enough to make the term $a\frac{r^n}{1-r}$ less than any assigned small quantity: if then n be infinite, this term becomes $=0$, and we have $s=\frac{a}{1-r}$. (iv)

For example, the series

$$s = 1+\tfrac{1}{2}+\tfrac{1}{4}+ \dots \text{ to infinity},$$
$$= \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2.$$

154. If we divide a by $1-r$ by the common process of division as far as n terms of the quotient, we obtain as the result of this algebraical process

$$\frac{a}{1-r} = a+ar+ar^2+ \dots +ar^{n-1}+\frac{ar^n}{1-r}.$$

If r be >1, the remainder is negative and $>$ the whole series of terms preceding it, and becomes continually greater in absolute value as n is increased. For example, if $r=2$, $n=3$, the remainder $=-8a$, and the above equality becomes

$$-a = a+2a+4a-8a.$$

It is only when $r<1$ that the remainder diminishes indefinitely as n is increased indefinitely.

155. Similar remarks apply to the series

$$\frac{a}{1+r} = a-ar+ar^2- \dots +(-1)^{n-1}ar^{n-1}+\frac{(-1)^n ar^n}{1+r},$$

Exercises. No. 50.

Find the last term and the sum of the series:—

(1) 1, 3, 9, ... to 5 terms.

(2) 2, 4, 8, ... to 8 terms.

(3) 3, $\frac{3}{2}$, $\frac{3}{4}$, ... to 7 terms.

(4) 5, $\frac{5}{3}$, $\frac{5}{9}$, ... to 4 terms.

(5) 3, $-\frac{1}{2}$, $1\frac{1}{8}$, ... to 4 terms.

Find the value of the following series to infinity:—

(6) 3, 1, $\frac{1}{3}$, ... (7) 2, $\frac{1}{2}$, $\frac{1}{8}$, ...

(8) $\cdot\dot{3}$. (9) $\cdot\dot{2}0\dot{7}$. (10) 1, $-\frac{1}{4}$, $\frac{1}{4}$, ...

CHAPTER IX.

ON EQUATIONS.

(1) On Finding Unknown Quantities.

156. Any assertion in algebraical form by means of the sign = that two expressions are equal to one another is called an equality. If one expression can be shown to be actually equal to the other, letter for letter, by simply altering the arrangement of the symbols, the equality is called an equation of identity, or simply an identity. An identity is true for all particular values which can be assigned to the symbols, like

$$a^2-b^2=(a+b)(a-b).$$

157. If, on the other hand, the equality is only true for certain particular values of the symbols involved, it is said to be a conditional equation, or an equation of condition, and its common name is *an equation*. Our general business with equations is to discover the values of the symbol, which are unknown to us, that make the equation true, or, as it is called, *satisfy it*. This is called *solving* the equation. x, y, z are the letters commonly used for unknown quantities; but any letter may become an unknown quantity through some particular application of a formula.

158. At present we shall only require x for our unknown quantity, for we are only going to deal now with equations involving one unknown quantity, and only in the first power, as x, not x^2 or x^3. Such equations are said to be of the first *degree*, and are also called simple equations.

159. A simple equation will often appear to be of a higher degree at various stages of the operations performed upon it, but all higher powers must finally disappear. Hence, a simple equation is an equation which when arranged as an integral expression contains the first power of x only.

160. Simple equations are solved by operations which depend upon the following axioms:—

i. If equals be added to equals, the results are equal.

ii. If equals be subtracted from equals, the results are equal.

iii. If equals be multiplied by equals, the results are equal.

iv. If equals be divided by equals, the results are equal.

v. If equals be raised to the same power, the results are equal.

vi. Like roots of equals are equal.

161. We shall commence with examples of an extremely elementary kind.

$$\begin{aligned} x+2 &= 5 \\ -2 &= -2 \\ \hline x &= 3 \end{aligned}$$ is solved by Ax. ii.

Subtract 2 from each side, or add -2, which is the same thing. This gives the solution, $x = 3$.

If we generalize this equation, we may write it,

$$\begin{aligned} x+b &= a \\ \text{Add to each side } -b, \quad -b &= -b \\ \hline x &= a-b \end{aligned}$$

Similarly,

$$\begin{aligned} x-b &= a \\ \text{gives, by adding } +b, \quad +b &= +b \\ \hline x &= a+b \end{aligned}$$

162. These results are comprised under the following general principle, which is much used in the solution of equations: viz.

I. Any quantity may be transposed from one side of an equation to the other by changing its sign.

Again, suppose we have $\frac{x}{3} = 5$, the two sides being equals will remain equals if we multiply them by 3,

$$\therefore \quad x = 15.$$

And if $$3x = 15,$$

these equals will remain equals if divided by 3, giving

$$x = 5.$$

Hence we have the general principle :—

II. Every term of an equation may be multiplied or divided by the same quantity.

If $-x+a=b-c$

$c-b+a=x$

$-b+c=x-a$, and we have the principle,

III. Every term of an equation may have its sign changed without altering the condition.

But if the three last principles be applied to some of the terms of an equation only, others being passed over, error will result.

IV. If more convenient, we may evidently interchange the two sides of the equation, which are equal to each other. They are sometimes called the two members of the equation.

Principle II is most frequently applied in order to clear an equation of fractions, as

$$\frac{x}{2}-\frac{x}{3}+\frac{x}{12}+2=4.$$

In this case both sides are multiplied by the L. C. M. of the denominators, which is here 12 : the equation then becomes

$$6x-4x+x=24.$$

The next step is to collect the coefficients, giving

$$3x=24.$$

Dividing by 3, $x=8$.

163. We may now describe the usual process in solving a simple equation as follows :—

i. Clear the equation of fractions.

ii. Transpose all terms containing x to the left-hand side of the equation, and all others to the right-hand side.

iii. Collect the coefficients of x.

iv. Divide both sides by the complete coefficient of x.

The order of i. and ii. may be sometimes with advantage interchanged.

164. But the whole process should be conceived in a more general manner, as follows :—

Suppose all the terms to be assembled, by transposition, if necessary, on the left-hand side of the equation. We have then

an expression, equated to 0, which contains x, and which we may consider to have been formed from x by a certain series of operations, as for instance, x may have been increased by 5, the result divided by 4, then increased by 5, then multiplied by 3, then diminished by 24. We shall undo all these operations by performing upon them in order the inverse operations: addition and subtraction being mutually inverse, and multiplication and division mutually inverse.

$$3\left(\frac{x+5}{4}+5\right)-24=0$$

Thus, in the example, the last operation was subtracting 24, accordingly we *add* 24: the previous operation was multiplying by 3, so we *divide* by 3: the one before, adding 5, so we subtract 5: before that, dividing by 4, so we multiply by 4: before that, adding 5, so we subtract 5, and this gives $x=7$, disentangled from all the operations in which it had been involved. The guiding principle then in the solution of equations is the successive performance of all the operations inverse to those indicated by the notation. And although it may sometimes be briefer to follow a slightly different order, the clear apprehension of this principle will always be found an advantage.

$$\begin{aligned}
3\left(\frac{x+5}{4}+5\right)-24 &= 0\\
+24 &= +24\\
3\left(\frac{x+5}{4}+5\right) &= 24\\
\frac{x+5}{4}+5 &= 8\\
-5 &= -5\\
\frac{x+5}{4} &= 3\\
x+5 &= 12\\
-5 &= -5\\
x &= 7
\end{aligned}$$

It is worth noting, that an equation must of necessity consist of unlike terms. If it were otherwise, if x entered as a multiplier into every term, the equation if true at all must be true always, and could afford no means of determining a particular value of x.

$$\frac{ax}{b}+\frac{cx}{d}=e.$$

Multiplying by bd, $adx+bcx=bde$;

$$(ad+bc)x=bde;$$

$$x=\frac{bde}{ad+bc}.$$

The use of collecting the terms is clearly to form a single term, which can be conceived as a single operation performed upon x, and so admit of an inverse operation.

165. In some cases we avoid high numbers by dealing with part of an equation and clearing it of fractions, and simplifying before the rest, especially when we can comprise two or more denominators under a simple L. C. M.

$$\frac{13x-5}{17}+\frac{2x-3}{4}+\frac{3-5x}{3}=\frac{6-7x}{12}.$$

Multiply by 12, the L. C. M. of 3, 4, 12,

$$\therefore \tfrac{12}{17}(13x-5)+3(2x-3)+4(3-5x)=6-7x;$$
$$\tfrac{12}{17}(13x-5)+6x-9+12-20x=6-7x;$$
$$\tfrac{12}{17}(13x-5)-7x=3;$$
$$156x-60-119x=51;$$
$$37x=111;$$
$$x=3.$$

This is often the best plan when there are fractions with compound denominators, if one of them occurs more than once;

$$\frac{1}{x-1}-\frac{2}{x+7}=\frac{1}{7(x-1)}.$$

Multiplying by $7(x-1)$,

$$7-14\frac{x-1}{x+7}=1;$$
$$6-14\frac{x-1}{x+7}=0;$$
$$3-7\frac{x-1}{x+7}=0;$$
$$3x+21-7x+7=0;$$
$$28=4x;$$
$$x=7.$$

166. When there are complex fractions, the first thing is to simplify them;

$$\frac{x-5\frac{1}{4}}{9}+\frac{15}{16}+\frac{14\frac{1}{4}-\frac{3x}{4}}{4}=60\tfrac{1}{4}-\frac{\frac{5x}{4}-24}{12}-2\tfrac{3}{4}x.$$

Multiply by 4,

$$\frac{4x-21}{9}+\frac{15}{4}+\frac{57-3x}{4}=241-\frac{5x-96}{12}-11x.$$

Multiplying by 36,

$$16x-84+135+513-27x=8676-15x+288-396x.$$

Transposing,

$$16x-27x+15x+396x=8676+288+84-135-513;$$
$$427x-27x=9048-648;$$
$$400x=8400;$$
$$4x=84;$$
$$x=21.$$

167. We now give some specimens of the solution of equations.

(1) $4x-11=29-x.$

By transposition, $4x+x=29+11;$

collecting terms, $5x=40;$

by division, $x=8.$

(2) $x+\frac{x}{3}+\frac{x}{15}=2x-3.$

Multiplying by 15, $15x+5x+x=30x-45;$

transposing, $-30x+15x+5x+x=-45;$

collecting terms, $-9x=-45;$

changing the signs, $9x=45;$

by division, $x=5.$

We should have done better in the last case, for the sake of avoiding negative signs, to transpose the known quantities to the right-hand side and the unknown to the left. We shall follow this method in the next example:—

(3) $5-\frac{x+4}{11}=x-3,$

$$55-x-4=11x-33,$$
$$55+33-4=11x+x,$$
$$84=12x;$$
$$\therefore\ x=7.$$

$$(4)\quad \frac{x-4}{x-6}=\frac{x+2}{x-4}.$$

Multiply by the L. C. M., viz. $(x-4)(x-6)$: we have

$$(x-4)^2=(x+2)(x-6).$$

$$x^2-8x+16=x^2-4x-12.$$

The term x^2 may be removed from each side,

$$\therefore\ -8x+16=-4x-12;$$

$$2x-4=x+3;$$

$$x=7.$$

Exercises. No. 51.

(1) $17x-3=45+11x.$

(2) $8x-5=13-7x.$

(3) $3a+x-5b+2=7b-a+c+b.$

(4) $2(x-3)=3(x-4).$

(5) $\frac{x}{2}-\frac{x}{3}+\frac{x}{4}=10.$

(6) $13\frac{3}{4}-\frac{x}{2}=2x-8\frac{3}{4}.$

(7) $\frac{49x}{3}+2=5x-14.$

(8) $\frac{5x}{2}-\frac{4x}{3}-13=\frac{5}{8}+\frac{x}{32}.$

(9) $2x+7+\frac{3}{2}x=6x-23.$

(10) $4-\frac{x+3}{6}=2+\frac{9-2x}{3}.$

(11) $\frac{5x}{7}-\frac{x}{3}-\frac{4x}{5}+\frac{7}{8}x=1-\frac{3}{4}.$

(12) $\frac{25}{x+3}=\frac{10}{3x-4}.$

(13) $12\frac{1}{4}+3x-6-\frac{7x}{3}=\frac{3x}{4}-5\frac{3}{8}.$

(14) $\frac{5x}{6}-\frac{3x}{4}-6=\frac{13}{3}-3x+\frac{x}{2}.$

(15) $\frac{21-5x}{5}+\frac{28-2x}{2}=\frac{8}{5}-x.$

(16) $\frac{x+1}{2}+\frac{x+2}{3}=16-\frac{x+3}{4}$.

(17) $\frac{x-3}{2}+\frac{x}{3}=20-\frac{x-19}{2}$.

(18) $\frac{3x}{5}-\frac{7x}{10}+\frac{3x}{4}-\frac{7x}{8}=-15$.

(19) $5x-\frac{x}{2}-282=\frac{x}{3}+\frac{x}{4}$.

(20) $18+\frac{36}{125}+\frac{63x}{50}=15x+\frac{3x}{2}$.

(21) $\frac{7x}{4}-12-\frac{1}{7}=\frac{15}{7}(x-4)-3x$.

(22) $\frac{3x}{49}-\frac{x-2}{9}=\frac{2x-49}{45}-\frac{x}{28}$.

(23) $\frac{16}{21}(x-5)+15=\frac{7x}{5}+3x-\frac{9}{7}$.

(24) $\frac{37x}{6}-\frac{3x}{8}+\frac{317}{2}=\frac{49x}{3}+19$.

(25) $12+\frac{1}{4}+3x-6-\frac{7x}{3}=\frac{3x}{4}-5\frac{3}{8}$.

(26) $\frac{3a+x}{x}-5=\frac{6}{x}$.

(27) $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=7x-712+\frac{x}{5}$.

(28) $3ax+\frac{a}{2}-3=bx-a$.

(29) $3\frac{2}{3}-x-\frac{9x}{2}+8=17-\frac{3x}{5}+\frac{3x}{2}$.

(30) $(3+x)(5+x)-48=\frac{99}{5}+x^2$.

(31) $\frac{1+x}{4}-\frac{21-5x}{30}+3-\frac{9x}{4}=\frac{x}{15}+\frac{x}{6}$.

(32) $\frac{cx^m}{a+bx}=\frac{fx^m}{d+ex}$.

(33) $3{\cdot}25x-5{\cdot}007-x=0{\cdot}2-0{\cdot}34x$.

(34) $13{\cdot}2x-\frac{3x}{4}+7{\cdot}6953=\frac{x}{5}+7834{\cdot}5$.

(35) $\frac{ex}{f}+\frac{cx}{d}+\frac{ax}{b}-g=h.$

(36) $\frac{ace}{d}-\frac{(a+b)^2x}{a}-bx=ae-3bx.$

(37) $\frac{3abc}{a+b}+\frac{a^2b^2}{(a+b)^3}+\frac{(2a+b)b^2x}{a(a+b)^2}=3cx+\frac{bx}{a}.$

(38) $48x^2+28x-48-(36x^2+36x-16)$
$=12x^2-13x+3.$

(39) $\frac{7x^n}{x-1}=\frac{6x^{n+1}+x^n}{x+1}-\frac{3x^n+6x^{n+2}}{x^2-1}.$

(40) $\frac{1}{x-2}-\frac{1}{x-3}=\frac{1}{x-4}-\frac{1}{x-5}.$

(41) $\frac{1}{x-a}-\frac{1}{x-b}=\frac{1}{x-c}-\frac{1}{x-d}.$

(42) $\frac{3}{x-2}-\frac{5}{x+2}=\frac{2x^2}{x^2-4}-2.$

(43) $\frac{4x+2}{x-3}-\frac{x+3}{x+1}=\frac{6}{x^2-2x-3}+3.$

(44) $\frac{2x-7}{x+5}-\frac{3x+5}{x-2}=\frac{4}{x^2+3x-10}-1.$

(45) $\frac{ax-b}{x+e}-\frac{cx-d}{x-e}=\frac{5}{x^2-e^2}+a-c.$

(2) Problems producing Simple Equations.

168. Arithmetical questions are usually solved by certain rules, which direct us to operate upon a given number by the simple processes of Addition, Subtraction, Multiplication, and Division, separately or in combination.

Each rule is equivalent to some such form as

$$\text{Answer} = 530 \times 31 \div 26 + 6 = 626;$$

or, if we employ general symbols, a, b, c, &c. for the data, and x for the unknown answer, we have some such form as

$$x = a \times b \div c + d,$$

where x appears by itself on one side of the sign of equality.

Here x, not being wrapt up with any of the data, nor itself to be operated upon, is said to be given explicitly (unfoldedly).

169. We have seen that an unknown quantity x may be determined by an equation under which it is wrapt up or folded in with the data (or even with itself), as in

$$\frac{x-1}{x-2}+\frac{x-3}{x-4}=2,$$

so as to be only separable by a series of processes. It is then said to be given implicitly. The equation may be put into words, as in Exercises, No. 44, and then it appears that such an equation is the algebraic expression of an implicit condition, i.e. a condition which makes the unknown quantity itself the subject of operation.

Questions that require the discovery of an unknown quantity determined by an implicit condition are called in Algebra Problems.

170. Some arithmetical questions are occasionally proposed in an *inverse* form (see *Scholar's Arithmetic*, chap. VII), naming the result of certain operations upon an unknown number, and demanding what number it is. These questions involve an apparent difficulty, viz. that we cannot in Arithmetic operate upon an unknown number; but our knowledge of the operations of Addition and Subtraction, or of Multiplication and Division, &c., as mutually inverse to each other, enables us to evade the difficulty, by transforming the questions into others, to which the rules of Arithmetic immediately apply. Thus the question, What number is that which increased by 5 becomes 11? is transformed into, What is the result of subtracting 5 from 11? just as in Algebra we transform the equation $x+5=11$ into $x=11-5$. Or again, What number is that which multiplied by 13 becomes 91? is transformed into, What is the result of dividing 91 by 13? just as $13x=91$ gives us $x=91\div 13$.

171. This process can be carried much further. Its principle is that upon which the solution of equations depends; see Art. 164. When more complex operations are involved, recourse should be had to Algebra, which by its general symbols greatly extends our power of expressing operations upon an unknown quantity, and thus of discovering by organized methods the unknown quantity.

We confine ourselves at present to problems producing simple equations.

172. Every problem divides itself into two parts :—

i. *The algebraic statement*, or expression of the condition in the form of an equation.

ii. *The solution of the equation.*

The solution of simple equations has been sufficiently considered in the last section.

We now deal with the algebraic statement.

173. The first thing is to fix on the unknown quantity, and to say plainly what it is, and what is its unit. As a general rule, it is the number demanded.

Consider, What does the question demand? It says in some way or other, How many? How many *what*? If the demand is for a number, the case is plain. Say, let x be the number required. If it is a length, or a magnitude of any kind, turn 'How much' into 'How many' by fixing on a unit, so as to be able to say, Let x be the number of feet required, or the number of ounces, or, if money, the number of shillings, or pence, or pounds, as most convenient. Then proceed to express in algebraical language the various operations performed on x, and finally express the condition.

This condition must be expressed as an equation, i.e. there must be two things, which we obtain separately and which we equate.

174. The simplest kind of condition is when the result of a certain series of operations upon x, which may be regarded as one compound symbolical operation made up of several, is stated to result in a given value.

Here we have, first, a symbolical result, and second, a numerical result, and we equate them.

For example, What is that number, the double of which, increased by 13, amounts to 57?

Take the clauses separately:—

What is that number,	Call it x,
the double of which,	viz. $2x$,
increased by 13,	i.e. $2x+13$,
amounts to 57?	$= 57$.

The result symbolically expressed is $2x+13$: but the last clause states it to be 57. We express the equivalence of these two forms of the result by equating them.

Another example. What is that number, the half of which, together with its third, is equal to 30?

What is that number,	Call it x,
the half of which,	viz. $\frac{x}{2}$,
with its third,	$+\frac{x}{3}$,
is equal to 30?	$\therefore \frac{x}{2}+\frac{x}{3}=30.$

In each case we complete the statement of all the operations upon x, or of the compound symbolical operation upon x, and equate the result thus expressed to the *given result.*

The same problem might be stated in many other ways, as—*A* has twice as much money as *B*, and three times as much as *C*, and *B* and *C* together have 30*s.*: how much has *A*?

175. More commonly, we have two symbolical results, derived from two compound symbolical operations, and these are stated to be equal to each other, or to be in some other way related to each other. For instance,

Example 1.—What is that number, the half of which, together with its third, increased by 5, is equal to twice the number, less 30?

What is that number,	Call it x,
the half of which,	viz. $\frac{x}{2}$,
with its third,	$+\frac{x}{3}$,
increased by 5,	$+5$.
First result, expressed symbolically,	$\frac{x}{2}+\frac{x}{3}+5.$
Twice the number,	$2x$,
less 30,	$-30.$
Second result, expressed symbolically,	$2x-30.$
Equation:	$\frac{x}{2}+\frac{x}{3}+5=2x-30.$

Example 2.—Two persons, A and B, start on a journey with a certain sum of money between them. A takes out 15*s.* and one-sixth of the rest: B then has 12*s.* and one-fourth of the rest, when they are found to have shared equally; how much was there at first?

Let x = the number of shillings at first.

A takes 15*s.*, there remains $x-15$, and the sixth part of this is $\frac{1}{6}(x-15)$, so that A has on the whole $15+\frac{1}{6}(x-15)$.

There remains $x-15-\frac{1}{6}(x-15) = \frac{5}{6}(x-15)$.

When B has had 12*s.*, there remains $\frac{5}{6}(x-15)-12$;

$$\therefore\ 12+\tfrac{1}{4}(\tfrac{5}{6}x-15-12) = 15+\tfrac{1}{6}(x-15).$$

We shall find $x = 159 = £7\ 19s.$

Example 3.—Divide a garden 40 feet wide so that there may be a gravel path on each side 4 feet wide, and a border between the path and the walls one-sixth of the width of the central portion.

We avoid fractions by taking x to be the width, in feet, of the border; then $6x$ is the width of the central portion; and

$$2x+2\times4+6x = 40,$$
$$8x = 32.$$

The border is 4 feet wide, and the central part 24 feet.

Example 4.—Three persons shared a ship's cargo. One person A had one-fourth, B had one-fifth, and C had £220. What were A's and B's shares?

Although A's and B's shares are the numbers asked for, yet we best avoid fractions by taking the value of the whole cargo in £'s, which is an unknown quantity $= x$.

Then A's share $= \frac{x}{4}$, B's share $= \frac{x}{5}$.

$$\frac{x}{4}+\frac{x}{5}+220 = x;$$

$x = £400$, A's share $= £100$, B's share $= £80$.

There are many similar cases in which it is convenient to take x to represent not the number demanded, but some other connected with it, and from which it may easily be derived.

Example 5.—A labourer is engaged for 50 days on these conditions: for each day he works he is paid three shillings and sixpence, but forfeits one shilling for every idle day; at the end of

the 50 days he is found to be entitled, under the agreement, to £6 19*s.*; required the number of days he worked and the number of idle days.

Let x = the number of working days.

$\therefore 50-x =$ „ „ idle „

$$\therefore \frac{7x}{2}-(50-x) = 139,$$

whence $x = 42,\ 50-x = 8.$

We shall now propose a question in a general form, i.e. involving general symbols only, which will of course apply at once to any particular case on the substitution of particular numbers for the general symbols.

Example 6.—A ship sails with a supply of biscuit for a days, at a daily allowance of m lbs. per head: after being at sea b days she encounters a storm, in which n of her crew are lost and damage sustained which will cause a delay of c days, and it is found that each man's share must be reduced to p lbs. a head. Find the original number of her crew.

Let x = the number demanded.

Then max = the number of lbs. of biscuit on sailing.

$m(a-b)x =$ „ „ „ b days later.

This is the first symbolical result.

After the storm the crew is $x-n$, and they have to remain at sea $a-b+c$ days, at a daily allowance of p lbs. per head.

In that time they will use up $p(a-b+c)(x-n)$, and this is the second symbolical result. But the two are equal, since they exactly consume the remainder.

$$\therefore m(a-b)x = p(a-b+c)(x-n),$$

$$x = \frac{np(a-b+c)}{(p-m)(a-b)+cp}.$$

Let $a = 60$, $m = 1$, $b = 20$, $n = 5$, $c = 24$, $p = \frac{5}{7}$, then $x = 40$.

176. Any question may be thus generalized by writing in the statement general symbols instead of particular numbers. For instance,

Example 7.—To divide the number 75 into two parts, so that the one shall be 4 times larger than the other.

Let x be the smaller part.

$\therefore$ $75-x$ is the larger.

But the larger is also $4x$;

$$\therefore\ 75-x = 4x;$$

$$x = 15,\quad 75-x = 60.$$

Instead of 75 write a, and instead of 4 write m, and we have the general question,

Example 8.—To divide the number a into two parts, such that one shall be m times as great as the other.

Let x be the smaller part.

$\therefore$ $a-x$ is the larger.

But this is also mx;

$$\therefore\ a-x = mx;$$

$$x = \frac{a}{m+1},\quad a-x = \frac{ma}{m+1}.$$

177. We now notice some cases in which the notions of ratio or proportion enter.

By what number must we multiply a to increase it in the ratio of $3 : 2$?

Let x be the multiplier; therefore it becomes ax, and

$$ax : a :: 3 : 2.$$

By the rule of extremes and means, it follows that

$$2ax = 3a,$$

$$x = \tfrac{3}{2},$$

the fraction which expresses the ratio $3 : 2$. Similarly, the multiplier which diminishes a in the ratio $11 : 12$ is $\frac{11}{12}$, and generally, the multiplier which alters a number in the ratio of $a : b$ is $\frac{a}{b}$. This last phrase, 'altering a number in a given ratio,' must be used with caution, being sometimes used differently. (See *Scholar's Arithmetic*, chap. XII. (5), note.) We call the fractions thus employed to modify numbers in a given ratio their 'Ratio-Multipliers,' or, where the same multipliers apply, as in Interest or Percentage, to a whole class of cases, 'Fixed Multipliers.' Most of these cases are best dealt with by finding at once their Fixed Multipliers.

Required to divide a number a into two parts, which shall be in the ratio of $m : n$.

Let x be one part, then $a-x$ is the other, and

$$x : a-x :: m : n;$$

$$\therefore \quad nx = m(a-x),$$

$$x = \frac{m}{m+n}a, \quad a-x = \frac{n}{m+n}a.$$

Thus these parts are found by multiplying the number by ratio-multipliers which have the terms of the ratio for numerators and their sum for denominators. This rule is very often useful. We think of it thus: the numbers which represent the ratio may represent the parts themselves, provided their sum represents the whole. This rule applies equally if there are several parts.

Required to divide x shillings between two boys in the ratio of their ages, which are 8 and 12.

Ratio-multipliers $\frac{8}{8+12} = \frac{2}{5}$ and $\frac{12}{8+12} = \frac{3}{5}$.

$\therefore$ the parts are $\frac{2x}{5}, \frac{3x}{5}$.

178. When a problem contains two or more unknown numbers which are stated to be proportional to given numbers, the unknown numbers are best expressed as corresponding multiples of x, which is taken for the prime multiple.

For instance, if we have to find three numbers, in the proportion of 3, 5, 7, which shall together make 315: we assume $3x$, $5x$, $7x$ to represent the numbers, and we have the condition,

$$3x+5x+7x = 315,$$

whence $x = 21$, and the three numbers are 63, 105, 147.

179. This mode of expression is useful in problems respecting movement, where two or more different rates of velocity enter. We usually take the number representing the slowest rate of movement, whether so many miles per hour, or feet per minute, &c., as the value of x.

But in problems respecting the movements of the hands of a watch, it is more convenient to take x to represent the number of spaces (signifying minutes) passed over by the minute-hand.

Then, since the minute-hand moves over 60 such spaces while the hour-hand moves over 5, the number moved over by the hour-hand in the same time $= \frac{x}{12}$.

For example, Required the time when the hands are together between three and four o'clock.

At three o'clock the minute-hand pointed to 12, since which it has moved through x minute-spaces. Meanwhile the hour-hand has moved through $\frac{x}{12}$: it had also a start of 15 minutes: when they are together,

$$x = \frac{x}{12} + 15;$$

$$x = \tfrac{12}{11} \times 15 = 16\tfrac{4}{11} \text{ minutes past three.}$$

180. To state the same question generally, Find the time when the hands are together between m and $\overline{m+1}$ o'clock.

At m o'clock the minute-hand pointed to 12, since which it has moved through x, and the hour-hand has moved in the same time through $\frac{x}{12}$, and it had also a start of $5m$ minutes,

$$\therefore \quad x = \frac{x}{12} + 5m,$$

$$x = \frac{12}{11} \times 5m = \frac{60m}{11}.$$

The interval between the hands being together (or their conjunction) between m o'clock and $\overline{m+1}$ o'clock and in the next hour is

$$60 + \tfrac{60}{11}\overline{m+1} - \tfrac{60}{11}m = 60 + \tfrac{60}{11} = 60 + 5\tfrac{5}{11},$$
$$= 65\tfrac{5}{11} \text{ minutes.}$$

It is therefore a fixed period, viz., the 11^{th} part of 12 hours

If the time of the hands being opposite to one another had been demanded, we should have had

$$x = \frac{x}{12} + 5m + 30,$$

and in a similar way any like problems may be treated.

181. A similar method applies also to the revolutions of the heavenly bodies about the earth or about the sun.

For instance, Given that the sun's apparent revolution about the earth is n days, and that of the moon about the earth in its orbit m days; find the length of a lunar month from new moon to new moon.

Think of the moon as the minute-hand, and of the sun as the hour-hand of a watch, and both starting from 12 o'clock.

Let x be the number of spaces, each equal to one day's motion, through which the moon has moved from the place of conjunction (new moon), then the sun will have moved through $\frac{mx}{n}$ such intervals in the same time; then, since m of such intervals brings the moon quite round again to the same place, and it has to go a little further, through the space just named, again to overtake the sun,

$$\therefore \quad x = m + \frac{mx}{n};$$

$$\therefore \quad nx - mx = mn, \qquad x = \frac{mn}{n-m}.$$

The real value of n is 365·25 nearly; and $m = 27{\cdot}32$ d.; and this gives $x = 29{\cdot}52$ d., or 29½ days nearly.

182. Success in stating problems greatly depends upon facility in the algebraical expression of quantities and their relations. The scholar should therefore practise himself in such exercises as the following.

Exercises. No. 52.

I. If x denote a sum of money in shillings, express—

(1) Its double, less three shillings.

(2) Half its value, less half-a-crown.

(3) The number of £'s in its third part.

(4) The number of pence left after paying a shilling away.

(5) One man's share, when three times the amount increased by five shillings is divided among nine men.

(6) A boy's share, when the amount increased by two pounds is divided equally between three boys.

(7) If the amount buys 10 loaves, find the price of a loaf.

(8) The charge per mile, if a railway ticket for 33 miles cost x.

(9) The charge for 11 miles at the same rate.

II. If x denote the number of years of a man's age, express—

(1) His age five years ago.

(2) His age five years hence.

(3) The present age of his children, who were born at intervals of two years and the youngest seven years ago.

(4) The age of his wife, who was two-thirds of his age at the time of their marriage ten years ago.

III. If x denote a length, expressed in yards, find

(1) Its half diminished by an inch.

(2) The remainder from 111 feet after deducting three times the said length.

(3) The remainder, after taking from it one-half of its length and one inch and one-fifth of the remainder.

(4) How to divide it into two parts in the ratio of 1 to 11.

(5) The time taken to traverse it at the rate of b yards a minute.

(6) The time that a train, b yards long, would require to pass another at rest, x yards long, at the rate of a miles an hour.

(7) To pass the same train, if the latter be moving at the rate of c miles an hour; first, in the same direction; secondly, in the opposite direction.

IV. The rate of movement of a train being x miles per hour, find—

(1) How far it has gone in 20 minutes.

(2) How far in a hours.

(3) In the time that a man who walks a miles in an hour takes to walk a mile.

(4) In the time that the same takes to walk b miles.

(5) How long does it take to go a mile?

(6) How long does it take to go b miles?

(7) How long does it take to go the distance which a man who walks a miles in an hour goes in b hours?

(8) What is its rate per minute in miles?

(9) What is its rate per minute in feet?

(10) What is the rule to express the rate of any moving body?

(11) What is the rate of a railway train which takes 40 minutes to go 30 miles?

(12) The train being a yards long, find the time it takes to pass, first, a given post; second, a train b yards long at rest; then the same moving at double its rate in the opposite direction.

V. (1) Express x per cent. of a.

(2) Express x per cent. of a if x per cent. of b is c.

(3) What must a tradesman charge for an article which cost him c, first, in order to make 5 per cent.; and secondly, in order to make d per cent. profit on it?

(4) What must a tradesman add to the price of an article c, first, in order to allow for a discount of 5 per cent.; secondly, of d per cent.?

(5) If x gallons of wine be taken out of a vessel of wine, containing a gallons, and the vessel be again filled up with water, find the quantity of wine in b gallons of the mixture?

Exercises. No. 53.

(1) A boy was sent to the Post Office with half-a-sovereign to procure a money-order, and was desired also to buy stamps to half the value of the money-order. He paid 1*d.* for the order, and brought back 2*s.* 2*d.* What was the amount of the order?

(2) A was sent on an errand to a neighbouring village; when he had been gone 8 minutes B ran after him with a forgotten parcel, and, by going twice as fast as A, overtook him a mile from home: what was A's rate of walking?

(3) Make the last question general by putting a minutes instead of 8 minutes, b times as fast for twice as fast, and c miles instead of 1 mile. Having found the general solution, find the particular answer if $a = 10$, $b = 2$, $c = 1$.

(4) Find two such numbers that their sum may be 24, one of them being twice as great as the other.

(5) Find two such numbers that the one may be m times as great as the other, and that their sum may $= a$. From the general result find the answer if $m = 3$, and $a = 128$.

(6) Find two such numbers that the one may be four times as great as the other, their difference being 36.

(7) A sum of £315 is to be so divided between A and B that A's share may be to B's share as 2 to 7. How much does each receive?

(8) How much money have I in my pocket, if the fourth and fifth parts of the same together amount to 11*s.* 3*d.*?

(9) A person has 264 fourpenny and threepenny pieces, and $4\frac{1}{2}$ times as many of the former as of the latter. What do they amount to?

(10) The sum of £650 is to be divided between two brothers, but so that the elder receives four times as much as the younger. How much does each receive?

(11) Two persons on going into partnership find that their common stock amounts to £9675, whereof one contributes twice as much as the other. What was each man's capital?

(12) A party of twenty are travelling by railway. The full fare being 6*s.* 8*d.*, and children travelling half-price, find how many children were of the party, supposing £5 to have been paid for fares.

(13) An innkeeper agreed to furnish refreshments for a party of 50 at 2*s.* a head for men, and 1*s.* 9*d.* a head for women: the account came to £4 15*s.*; of how many men and women did the party consist?

(14) If 2 women can do the work of 3 boys, and if 3 men can do the work of 5 women; find a woman's share of £44, which is paid to 5 boys, 3 women, and 5 men for doing a piece of work.

(15) Divide £26 3*s.* 3*d.* between three persons, so that their shares may be to one another in the proportion of the fractions $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$.

(16) Four towns whose populations respectively amount to 1350, 2425, 3200, 5445, contribute according to population to make up the sum of £2732 8*s.*; how much is paid by each?

(17) A boy buys a certain number of oranges at 3 for 2*d.*, and one-third of that number at 2 for 1*d.*: at what price must he sell them to get 20 per cent. profit? If his profit be 5*s.* 4*d.*, find the number bought.

(18) At what time will the hands of a watch be together between eight and nine o'clock?

(19) At what time will the hands of a watch be opposite to each other between eight and nine o'clock?

(20) From the last question point out what is wanted to make the last paragraph of Art. 180 complete for all cases. Take for examples the intervals from four to five and from seven to eight o'clock.

(21) The eldest of four children receives from the property left by his father £1000 and the fifth part of what remains: after this the second receives £2000 and the fifth part of what remains: after this, the third receives £3000 and the fifth part of what remains, and the fourth receives what is then left. Supposing the shares to turn out equal, what was the amount of the property, and what did each get?

(22) Two persons, A and B, drink beer out of an eighteen gallon cask for eighteen days, when B leaves off, and A, by reducing his consumption by one quarter, finishes the cask in 12 more days; find the daily consumption of each, on the supposition that A's share of the cask is thrice that of B.

(23) A person invests £1700, part of it in 3 per cent. stock at 90, and the rest in $3\frac{1}{4}$ per cent. stock at 104, and obtains an income of £55 per annum: what sum does he invest in each?

(24) A man's age is three times that of his son; ten years ago it was five times: what is his age?

(25) Find a number such that if we add to it its half, the sum will be as much above 60 as the number itself is below 65.

(26) There are two stations, A and B, 1760 yards distant from each other. A man starting from A at two o'clock, and walking uniformly, reaches B at half-past two. Another man starting from B at ten minutes past two reaches A at twenty-five minutes past two. At what distance from A did the two men pass each other?

(27) A traveller sets out from A for B, and walks 3 miles per hour. Fifteen minutes afterwards another traveller sets out from B for A, and walks $3\frac{3}{4}$ miles per hour; and he goes two miles

beyond the middle point between B and A before he meets the first traveller. Find the distance between B and A.

(28) In a contested election B has $\frac{1}{3}$ of the votes polled: and the number of votes given for him bears to C's the ratio of 7 : 6. A has 2304. How many had B and C?

(29) An estate of 864 acres was divided amongst A, B, and C; A's share is to B's as 5 to 11, and that of C is equal to the sum of the other shares: find them.

(30) The sum of two numbers is 13, the difference of their squares is 39. What are they?

(31) In a half-mile walking match B's step was 5 inches shorter than A's, but B took 16 steps to 11 of A's and won by 155 yards; find A's step.

(32) Two men purchase an estate for £9000. A could pay the whole, if B gave him half his capital, while B could pay the whole, if A gave him one-third of his capital. How much money had each of them?

(33) Divide the number 208 into two parts, such that the sum of one-quarter of the greater part and one-third of the less, when increased by four, shall equal four times the difference of the two parts.

(34) A man had a number of pennies, which he wished to arrange in the form of a square. At his first attempt he had 130 over, but when he increased the side of the square by 3 pennies he had only 31 over. How many pennies had he?

(35) Two numbers are in the ratio 4 : 5. If the one is increased and the other diminished by 10, the ratio of the resulting numbers is inverted. Find the numbers.

(36) The difference between a certain number and its defect from 15 is 11; find the number.

(37) The difference between a certain number and its defect from 17 is equal to three times its defect from 21; find the number.

(38) There are two numbers which together make 60. If 5 be added to the larger number, and the sum be divided by 3, the quotient is half the smaller number. Find the difference of the two numbers.

(39) From a bag of sovereigns are taken out, first, one-tenth of

the whole, then 10 sovereigns, then one-seventh of the remainder, and after this the bag is found to have in it three-fifths of its original contents: how many sovereigns have been taken out?

(40) Saltpetre and sulphur in the proportion of 7 : 3 make up together a weight of 80 lbs. How much saltpetre must be added, or how much sulphur taken away, if the proportion of these two substances is to become 11 : 4.

(41) A man bought a certain quantity of apples to divide among his children. To the eldest he gives half the whole all but 8 apples; to the second he gives half the remainder all but 8 apples. In the same manner also does he treat the third and fourth children. To the fifth he gives 20 apples that remain. How many did he buy?

(42) From a sum of money I take away £50 more than the half, then from the remainder £30 more than the fifth, then from the second remainder £20 more than the fourth part: at last only £10 remain. What was the original sum?

(43) Two partners *A* and *B* possessing equal shares of a business agree to dissolve partnership. *A* takes one-tenth of the stock in trade, *B* takes the rest and pays £560 to *A*. What is the stock in trade worth?

(44) To find an arithmetical mean between 7 and 17?

(45) To find three arithmetical means between 11 and 17?

(46) A sum of £9 16*s.* 11*d.* was paid in 34 payments, which increased by 3*d.* a time; what was the first payment?

(47) At what times are the hands of a watch at right angles between five and six o'clock?

(48) A man starts to run when the hands of his watch are together, and stops as soon as they come into the same straight line. He finds that he has run 13,200 yards. How many miles per hour would he pass over at this speed?

(49) The hands of a watch were observed to be together when the clock struck twelve, and again at six minutes past one; what time will the watch indicate when the clock strikes three 10 days hence?

(50) If a man can row from *A* to *B* with the stream in 45

minutes and takes 75 minutes in coming back, compare the rate at which he rows with the rate of the stream.

(51) A man walking at the rate of 4 miles an hour observes that a train 88 yards long passes him in 5 seconds; at what rate is it moving?

(52) A train 66 yards long, which had come 50 miles from the terminus in one hour, met another 110 yards long, which it passed in 5 seconds; at what rate was the latter moving, and where will it meet a goods train, moving at the rate of 20 miles an hour, which left the terminus half-an-hour after the first?

(53) A person has bought goods for £40, he sells a portion of them at a gain of 5 per cent. and the remainder at a gain of 35 per cent., and gains 30 per cent. on the whole transaction. Determine the two portions.

(54) A person sells goods which cost him £a in two portions, at a gain of m and n per cent. respectively. His gain on the whole being p per cent., determine the two portions.

(55) A buys a cask of 110 gallons of wine for £120, and sells half of it at 24s. a gallon. At what rate must he sell the rest in order to make 25 per cent. profit?

(56) A buys 1000 quills for 20s. and sells half at 2s. 6d. a hundred; what must he charge per 100 for the remainder to make 50 per cent. by the whole?

(57) A person walked to the top of a mountain at the rate of $2\frac{1}{3}$ miles an hour, and down the same way at $3\frac{1}{2}$ miles an hour, and was out 5 hours; find the distance to the summit.

(58) A man has a sum of money amounting to £23 15s., consisting of half-crowns and florins; he has in all 200 pieces of money: how many has he of each sort?

(59) Two shopkeepers went to the cheese fair with the same sum of money. The one spent all his money but 5s. in buying cheese, of which he bought 250 lbs. The other bought at the same price 350 lbs., but was obliged to borrow 35s. to complete the payment. How much had they at first?

(60) A person invests £1024 at a certain rate of interest, and invests the dividend in the same way. At the end of two years he

receives back for his capital and compound interest on it the sum of £1156: what was the rate of interest?

(61) A cistern can be filled by two pipes together in $1\frac{1}{8}$ hour. The larger pipe by itself will fill it in one third of the time required by the smaller. What time will each pipe separately take to fill it?

(62) By his will a man leaves a sum of £2520 to be divided among his four sons, *A*, *B*, *C*, *D*, as follows: *C* is to have £360, *B* as much as *C* and *D* together, and *A* £1000 less than twice as much as *B*. How was the money divided?

(63) From the 8th to the 19th of June last year the thermometer was observed to ascend a degree and a half each day, and the mean of all the observations was 74°. What was the height of the thermometer on the 8th of June?

(64) With equal volumes of two fluids, one of which is three times as heavy as the other, form two mixtures of equal volume, one of which shall be twice as heavy as the other.

(65) If apples sold at four pecks for 3*s*. yield $16\frac{2}{3}$ per cent. profit, what will be the rate of profit per cent. when three pecks are sold for 4*s*.?

(66) A man bought a house which cost him 5 per cent. upon the purchase money to put it in repair. It then stood empty for a year, during which time he reckoned he was losing 4 per cent. upon his total outlay. He then sold it again for £1192, by which means he gained 10 per cent. upon the original purchase money. What did he give for the house?

(67) A person starts from Ely to walk to Cambridge (which is distant 16 miles) at the rate of $4\frac{4}{9}$ miles an hour, at the same time that another person leaves Cambridge for Ely walking at the rate of a mile in 18 minutes; where will they meet?

(68) A tradesman buys some stuff at the rate of 7 shillings for 5 yards; he sells it again at 11 yards for 16 shillings, and gains 24 shillings by the sale: how many yards were there?

(69) Three vessels, *A*, *B*, *C*, whose capacities are in the proportions of 2 : 3 : 4, each contain a quantity of spirits equal to half the content of *A*. *C* and *B* are filled with water. *A* is filled from *C*,

and then C is partly filled with unknown equal quantities from A and B. The spirits and water in C are found to be in the ratio of $2:5$. Find the quantities taken from A and B.

(70) In an examination three-fifths of full marks are represented by a number, with two consecutive digits, the former digit being the greater. The number which is formed by inverting the digits of this number represents half the full marks. What are they?

(71) The distance from London to Cambridge (viâ Hitchin) is 58 miles, and from Cambridge to Hitchin 26 miles. An express train from London to Cambridge (for which ten minutes is usually allowed at Hitchin) is detained there by an accident twenty minutes over the time, and afterwards proceeding only at three-fifths of its former speed arrives at Cambridge forty-five minutes late. Find the speed at first.

(72) In a gallons of water (each gallon weighing 10 lbs.) b lbs. of salt were dissolved; how much water must be added in order that there may be only g per cent. of salt, by weight?

(73) A farmer calculated that he had sufficient food to support p cows through the winter. After feeding them 9 days, the cattle plague broke out and carried off r of them daily for 10 days. In consequence also of a long continuance of frost he was obliged to feed the remaining stock three days longer than he had expected, when the food was exhausted. How many days would the food have lasted had no deaths occurred?

(74) A steamboat, leaving Dover with a favourable wind, arrives at Calais in two hours. On its return, the wind being contrary, it made 6 miles an hour less than during the former voyage. When it was halfway across, the wind changed and increased its speed by 2 miles an hour, in consequence of which the whole time of the passage was diminished, compared with what it would otherwise have been, in the ratio of 6 to 7. Find the distance from Dover to Calais, and the two rates of speed on the second voyage.

(75) Required the contents of three casks, which are such, that when the first is filled from the second there still remains two-ninths in the latter; whilst if the second be filled from the third,

there remains in the last one-fourth; finally, the first being emptied into the third, 50 gallons are still wanted to fill it.

(76) Supposing the periods of the planet Venus and of the Earth in their orbits to be 224 days and 365 days respectively, find how often Venus is between the Earth and the Sun (in conjunction).

(77) The period of the planet Mercury in its orbit being 88 days, find the period between two conjunctions of the Earth and Mercury.

(78) The specific gravity of milk is $1\frac{1}{32}$. Twenty gallons of milk, being supposed to be adulterated with water, were weighed and found to weigh 205 lbs. Supposing a gallon of water to weigh 10 lbs., find the quantities of water and milk.

(79) The specific gravity of lead is 11·324, that of cork ·24, and that of deal ·45; it is required to form of lead and cork a mass weighing 80 kilograms, to weigh as much as an equal volume of deal. Find the quantities of lead and cork.

(80) A certain wine, at m shillings per gallon, is to be mixed with another, at n shillings per gallon; what will be the proportions, so as to sell the mixture at p shillings per gallon?

(81) If the work done by $x-1$ men in $x+1$ days is to the work done by $x+2$ men in $x-1$ days in the ratio of 9 : 10, find the number of men and the number of days in each case.

(82) If m women can do the work of n boys, and if p men can do the work of q women, find a woman's share of £100, which is paid to c boys, b women, and a men for doing a piece of work.

(83) In a race over a course of l yards A can give B a yards start, and B can give C b yards start. How many can A give C? and if B is placed a yards and C $(a+b)$ yards in advance of A, find how far A will have to run to overtake C.

(84) P and Q are two places 12 miles apart: A travelling uniformly leaves P at the same moment that B and C leave Q travelling in opposite directions, C's rate being 5 miles per hour; when A meets B, C is 9 miles distant from both. What must be A's rate of travelling that he may overtake C at the same instant that B enters P?

(3) Equations containing Two Unknown Quantities.

183. If we have an equation of the form $2x+3y=13$, containing two unknown quantities, x and y, we can satisfy it by an infinite number of pairs of values of x and y, supposing that they may be perfectly general, i. e. that they are not restricted to be integral or fractional, positive or negative. For on these suppositions every simple equation has a root. If then we were to assign to y any particular value, as 2, we should have a simple equation of one unknown quantity, viz. $2x+6=13$, which will necessarily have a solution, and so yield a value of x to satisfy it. But the number of values that may thus be assigned to y is unlimited, and whatever y may be, there is a corresponding value immediately to be determined for x. So, on the other hand, x may be anything whatever, and we shall, as before, be able to determine a corresponding value for y.

For example, if $y=2$, then $x=3\frac{1}{2}$;

if $x=2$, then $y=3$;

and these two pairs of values satisfy the equation. Again,

if $y=1$, then $x=5$;

if $x=1$, then $y=3\frac{2}{3}$;

and we might find as many more pairs of values as we please.

But now, suppose that we have a second equation, as

$$3x+2y=12,$$

which is also to be satisfied. Although this latter equation, taken by itself, might also be satisfied by an infinite number of pairs of values of x and y, yet it will be found that only *one pair* of these values will satisfy both equations *simultaneously*.

I. This pair of values may be determined by the following method, called the Method of Elimination. It consists in eliminating between the two equations (that is, getting rid of) one of the unknown quantities, so that there remains a simple equation containing the other unknown quantity together with known quantities.

If necessary, clear the equations of fractions, and arrange them under each other, like terms under like, as

$$3x+2y=12,$$
$$2x+3y=13.$$

Multiply the first equation by the coefficient of y in the second, and the second by the coefficient of y in the first; then

$$9x+6y=36,$$
$$4x+6y=26.$$

The terms containing y are now equal, and by subtracting we obtain an equation which contains x only, viz.

$$5x=10,$$
$$\therefore\ x=2.$$

We may substitute this value of x in either of the original equations to find y. Thus,

$3x+2y=12$ becomes
$$6+2y=12,$$
$$2y=6,$$
$$y=3;$$

$x=2$, $y=3$ are the required pair of values.

184. This is the simplest way of treating the equations, so as to have no fractions: but the readiest way of seeing the reason of the rule is perhaps to divide first by the coefficients of y, so as to have y alone in each equation, which will of course disappear on subtraction, as

$$ax+by=c, \qquad \frac{ax}{b}+y=\frac{c}{b},$$
$$a'x+b'y=c'; \qquad \frac{a'x}{b'}+y=\frac{c'}{b'};$$

and then to divide each equation by the coefficient of x, so that x may appear alone in each, as

$$x+\frac{by}{a}=\frac{c}{a},$$
$$x+\frac{b'y}{a'}=\frac{c'}{a'};$$

giving $$\left(\frac{a}{b}-\frac{a'}{b'}\right)x=\frac{c}{b}-\frac{c'}{b'}, \qquad x=\frac{cb'-c'b}{ab'-a'b};$$

$$\left(\frac{b}{a}-\frac{b'}{a'}\right)y=\frac{c}{a}-\frac{c'}{a'}, \qquad y=\frac{ca'-c'a}{a'b-ab'};$$

giving the same rule as before. The method is universally applicable, if we pay the proper attention to the signs.

II. Another method frequently employed is that of *substitution*. This consists in finding the value of one of the unknown quantities in terms of the other from one equation, and substituting the expression so found for the first quantity in the other equation.

Example. $$\left.\begin{aligned} 3x+2y&=121,\\ x&=3y. \end{aligned}\right\}$$

Substituting the value $3y$ for x in the first equation, we have

$$3\times 3y+2y=121,$$

or $$9y+2y=121,$$

$$11y=121,$$

$$\therefore\ y=11.$$

Generalizing, $$\left.\begin{aligned} ax+by&=c,\\ mx&=ny. \end{aligned}\right\}$$

Here $$x=\frac{ny}{m}, \qquad \therefore\ \frac{na}{m}y+by=c,$$

$$\frac{na+mb}{m}y=c;$$

$$\therefore\ y=\frac{mc}{na+mb},$$

and $$x=\frac{nc}{na+mb}.$$

III. A third method is frequently given, that *of comparison*, which consists in determining the value of one of the unknown quantities in terms of the other from each equation, and equating them. This is only a form of substitution, and need not be considered separately.

Exercises. No. 54.

(1) $3x+7y=8$,
$4x+9y=11$.

(2) $3x=5y$,
$x+y=7$.

(3) $x+y=13$,
$x-y=2$.

(4) $3x+2y=118$,
$x+5y=191$.

(5) $2x-5y=4$,
$5x+3y=41$.

(6) $\frac{x}{3}+\frac{y}{2}=7$,
$\frac{x}{4}-\frac{y}{3}=1$.

(7) $8x-15y=\frac{x}{2}$,
$2x+y=15$.

(8) $3y-7x=4$,
$2y+5x=22$.

(9) $11x-10y=14$,
$5x+7y=41$.

(10) $2x+3y=17$,
$5x-3y=11$.

(11) $4x+3y=24$,
$5x-2y=7$.

(12) $2x+5y=20$,
$3x-4y=7$.

(13) $60x-17y=146$,
$48x+13y=170$.

(14) $7x+\frac{5y}{2}=411\frac{1}{2}$,
$39x-14y=-935\frac{9}{10}$.

(15) $5x-8\frac{1}{2}=7y-44$,
$2x=y+\frac{5}{7}$.

(16) $7y=2x-3y$,
$19x=60y+621\frac{1}{4}$.

(17) $5\frac{3}{4}y-11x=4y+117\frac{1}{8}$,
$8x+175=2y$.

(18) $3x+5y=\frac{105}{4}$,
$49x-\frac{189}{5}+48y=9x+273$.

(19) $13x+7y-341=7\frac{1}{2}y+43\frac{1}{2}x$,
$2x+\frac{y}{2}=1$.

(20) $6x=3y-4$,
$4y+\frac{7\times19}{2\times3}=\frac{16}{3}+27x$.

(21) $113\frac{1}{2}x-27\frac{5}{7}y=10y+5488\frac{4}{7}$,
$9y-347=5x-420$.

(22) $\dfrac{3}{4+y}=\dfrac{4}{9+x}$,

$3x+8y=11$.

(23) $x+y=18{\cdot}73$,

$0{\cdot}56x+13{\cdot}421y=763{\cdot}4$, to 3 places.

(24) $(x+5)(y+7)=(x+1)(y-9)+112$,

$2x+10=3y+1$.

(25) $x+y=30$,

$10x-11y=486$.

Remark.—Equations sometimes contain, instead of x, y, the reciprocals of these quantities, in such a manner that they may be solved with respect to these quantities (the reciprocals) as they stand, and the values of x and y derived from them afterwards.

(26) $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{8}{15}$,

$\dfrac{1}{x}-\dfrac{1}{y}=\dfrac{2}{15}$.

(27) $\dfrac{7}{x}-\dfrac{5}{y}=\dfrac{1}{6}$,

$\dfrac{3}{x}+\dfrac{4}{y}=1\frac{2}{21}$.

(28) $\dfrac{m}{x}+\dfrac{n}{y}=a$,

$\dfrac{m}{y}+\dfrac{n}{x}=b$.

(29) $\dfrac{3}{x}+\dfrac{2}{y}=2\frac{1}{6}$,

$\dfrac{4}{x}+\dfrac{5}{y}=3\frac{1}{3}$.

(30) $\dfrac{1}{x}-\dfrac{4}{y}=\dfrac{1}{18}$.

$18x+10y=7xy$.

(31) $\dfrac{m}{x}-\dfrac{n}{y}=a$,

$px=qy$.

(4) Problems producing Simultaneous Equations.

185. When two unknown quantities are sought, in order to determine them we must have two conditions, and these, expressed in Algebraic language, give two equations, to be satisfied simultaneously.

In several of the problems solved by means of simple equations two unknown quantities were demanded. They were, however, so connected that one of them could be immediately expressed in terms of the other. These problems may, of course, be solved

also by means of two simultaneous equations, and the statements are often made with more facility by the employment of two unknown quantities. For example,—

Example 1.—Required to find two numbers, the sum of which is 29, and which are such that three times the less exceeds twice the greater by 2.

Call the greater number x, the lesser y,

then $$x + y = 29,$$
$$3y - 2x = 2.$$

Doubling the first and adding, we have

$$3y + 2y = 58 + 2,$$
or $$5y = 60,$$
$$y = 12, \quad x = 17.$$

Example 2. A number consisting of two digits, when increased by 27, has the same digits, but in different order; and one digit is two-thirds of the other. Find the number.

Let x be the tens' digit, y the units' digit of the number;

$\therefore$ $10x + y$ represents the number, and

$$10x + y + 27 = 10y + x;$$
$$\therefore \quad 9y - 9x = 27,$$
$$y - x = 3.$$

Thus $y > x$; and we have $x = \frac{2}{3}y$,

$$\therefore \quad \tfrac{1}{3}y = 3,$$
$$y = 9, \quad x = 6.$$

The number is 69.

This problem might have been easily solved tentatively, without the second condition, owing to the fact that the values of x and y are limited to the 9 digits.

Example 3.—To find a fraction such that when 1 is added to its numerator its value becomes $\frac{1}{3}$, and when 1 is added to the denominator it becomes $\frac{1}{4}$.

Let x denote the numerator, y the denominator; then $\frac{x}{y}$ is the fraction.

Also $$\frac{x+1}{y}=\frac{1}{3}, \qquad \frac{x}{y+1}=\frac{1}{4};$$

$$\therefore \quad 3x+3=y,$$

$$4x=y+1;$$

$$\therefore \quad 4x=3x+3+1,$$

$$x=4, \quad y=15;$$

$$\therefore \quad \frac{x}{y}=\frac{4}{15}, \text{ the fraction required.}$$

Example 4.—If three lbs. of tea and two of coffee cost 11*s.*, whereas two lbs. of coffee and three of tea cost 9*s.* 10*d.*, find the price of tea and coffee per lb.

Let x be the price of 1 lb. of tea, in shillings;

Let y be the price of 1 lb. of coffee, in shillings.

Then $$3x+2y=11,$$

$$2x+3y=9\tfrac{5}{6};$$

which gives $$x=2\tfrac{2}{5}=2s.\ 8d.,$$

$$y=1\tfrac{1}{2}=1s.\ 6d.$$

186. The above example is a specimen of an important class of problems, of wide application in physical subjects, in which we are required to separate two ingredients of a compound body, by means of their specific properties, such as their different specific gravities. Two experiments, as it were, are necessary, giving two facts respecting the body, for instance, (i) its *volume*, (ii) its *weight*. These will furnish two equations.

Example 5.—Nine gallons of a mixture of fresh and sea water are found to weigh 91 lbs. 4 oz.; assuming that a gallon of fresh water weighs 10 lbs., and that the specific gravity of sea water is 1·025, find the quantity of each in the mixture.

Let x be the number of gallons of fresh water.

Let y be the number of gallons of sea water.

$$x+y=9,$$

$$10x+\frac{41y}{4}=91\tfrac{1}{4}, \text{ whence}$$

$$x=4, \quad y=5.$$

Example 6.—Hiero's crown is said to have weighed 20 lbs., which was 16 times the weight of an equal volume of water: on the supposition that it consisted only of gold (sp. gravity $= 19\frac{1}{4}$) and silver (sp. gravity $= 10\frac{1}{2}$); find the quantities of each metal.

Let $x =$ the number of lbs. of gold.

Let $y =$ the number of lbs. of silver.

$$\therefore \quad x+y=20;$$

$$\frac{x}{19\frac{1}{4}}+\frac{y}{10\frac{1}{2}}=\frac{20}{16};$$

since, if we convert separately the gold and the silver into the corresponding volumes of water, we diminish the weight (20 lbs.) in the ratio of 1 : 16.

Result. $x = 15$ lbs. 2 oz., $y = 4$ lbs. 14 oz.

Exercises. No. 55.

(1) To find two numbers whose sum is 120 and difference 18.

(2) To find two numbers whose sum is a and difference b.

(3) A and B had an equal unknown number of marbles; C had 10, which he shared at random between A and B; A then found that he had 19. and B had 23. What number had they at first, and what did each receive?

(4) A traveller in France had to receive 745 francs; he was paid entirely in 5 franc and 2 franc pieces, 185 in all. How many were there of each?

(5) Three men and two boys working together one day were found to have dug 39 feet of a garden border; on another day two men and three boys had dug 36 feet: compare the work of a boy and of a man.

(6) Two travellers, A and B, start on a journey together, A with £100, and B with £48. A spends twice as much as B, and at the end of the journey A has three times as much as B; how much has each spent?

(7) Find such a fraction that when 2 is added to its numerator its value becomes $\frac{2}{3}$, and when 5 is added to the denominator the value is $\frac{1}{4}$.

(8) Find such a fraction that when half the numerator is taken from the denominator its value becomes $\frac{1}{4}$, and that six times the numerator is to the denominator as 4 : 3.

(9) A servant, who was sent to purchase 5 lbs. of tea of one kind, and 2 lbs. of another, which were to cost 18*s.* 10*d.*, mistook between the two kinds, and had to pay 2*s.* more; how much did each cost a lb.?

(10) A number consists of two digits: if it be multiplied by 2, and the product diminished by 4, the digits are inverted; and if 10 be subtracted from it, the remainder is equal to three times the sum of the digits. Find the number.

(11) There are two towns, the joint population of which has increased 2 per cent. between the last two censuses. If each had increased at the same rate per cent. as the other has, the increase on the joint population would have been 5 per cent. Also the population of one of these towns is known to have been at the last census four times that of the other. Find the rates per cent. of increase of each of the towns.

(12) There are two chests of mixed tea, in one the green is mixed with the black in the ratio of 3 : 5, in the other in the ratio of 5 : 11. What number of pounds must be taken from each chest so as to form another mixture which shall contain exactly 19 lbs. of green tea and 37 lbs. of black?

(13) A man has two boxes, one containing £27 and the other £4 10*s.*, and also two bags of money. He puts one bag into each box, and then each box contains the same amount, but when he reverses the bags one box contains three times as much as the other: find the amount in each bag.

(14) Two sums of money are together equal to £54 12*s.*, and there are as many pounds in the one as shillings in the other: what are the sums?

(15) A person invests £1712, part of it in Three per Cent. Stock at 90, and the rest in Three and a Quarter per Cent. Stock at 104, and obtains an income of £55 per annum; what sum does he invest in each?

(16) Water is a compound of oxygen and hydrogen in the pro-

portion of eight parts of oxygen to one of hydrogen, estimated by weight; it is required to take from two closed vessels, containing respectively 19 lbs. weight of hydrogen and 117 lbs. weight of oxygen, such quantities as might be combined to form water and to leave in the vessels equal weights of oxygen and of hydrogen.

(17) Water is a compound of oxygen and hydrogen in the proportion of one part of oxygen to two parts of hydrogen, estimated by volumes at the same temperature and pressure, oxygen being 16 times as heavy as hydrogen. Supposing common air to contain twenty-one parts of oxygen to seventy-nine of nitrogen, by volume; find how the oxygen, hydrogen, and nitrogen contained at equal temperatures and pressures in closed vessels containing respectively 873, 1100, and 1127 cubic feet may be divided so as to furnish the constituents necessary to form water and air and to leave equal volumes in the three vessels.

(18) Two vessels contain each a mixture of wine and water. In the first vessel the quantity of wine : quantity of water : : 1 : 3, and in the second : : 3 : 5. What quantity must be taken from each in order to form a third mixture which shall contain 5 gallons of wine and 9 of water?

(19) A, B, C are three clocks. A gains 5 times as many seconds and C six times as many seconds in an hour as B loses minutes. Supposing all the clocks to be set right at noon, and that when C marks two minutes past 6 p. m., A is as many seconds behind C, as B wants minutes to 6; find the irregularities of A and B.

(20) A and B start at noon on March 1 to meet each other. A would travel over the distance between them in a year of 365 days; but A travels $6\frac{1}{2}$ hours per day, and B 7 hours, and B goes as far in $6\frac{1}{2}$ hours as A in 7 hours. Supposing each to travel uniformly, after how many days will they meet?

(21) Given that two-thirds of a mass of iron weighs 96 lbs. more than three-fourths of a certain block of stone, and that five-eighths of the latter weighs precisely as much as four-ninths of the former: what do the two masses weigh?

(22) Three countries A, B, C. make use of three different

measures for cloth; 15 measures of A together with 33 of B are equivalent to $39\frac{1}{2}$ of C; again, 24 of A and 55 of B are together equal to 65 of C: required the ratios of A and B to that of C, and find by how much per cent. A and B differ.

(23) The units' digit in a number less than 100 is three times the tens' digit. If a second number be formed having for its digit in the units place the *sum* of the digits of the first number, and for its digit in the tens place their *difference*, it is found that the second number exceeds the first by 22: find both numbers.

(24) Find a fraction such that, if 1 be taken from its numerator and added to its denominator, it becomes $\frac{1}{2}$; but if 1 be taken from its denominator and added to its numerator, the result is $\frac{4}{5}$.

(25) A and B enter into partnership with unequal sums of money. They agree that each shall receive interest on his capital at the rate of 4 per cent. per annum, and that all subsequent profits shall be equally divided. At the end of the first year they receive £428 and £508 respectively; and it is found that A thus receives £77 more than he would have had if they had shared in proportion to the capital invested by each. Find each man's capital.

(26) The sum of the first three terms of an arithmetic progression is 9, and the sum of the three subsequent terms is 27; find the series.

(27) A merchant at Bordeaux has wine of two qualities; he finds he can sell a mixture of 3 hectolitres of the better kind with 5 of the inferior at 20 fr. 50 c. the hectolitre; and if he mixed $3\frac{3}{4}$ of the better with $7\frac{1}{2}$ of the inferior, he could sell the hectolitre at 20 francs; what did each kind cost him?

(28) A mass of silver alloyed with copper, weighing 1480 lbs., has the same volume as $14\frac{2}{3}$ gallons of water: how much is there of each metal, taking the specific gravity of silver as 10·5, and that of copper as 9?

(5) Quadratic Equations.

187. An equation which, when reduced to its simplest form, contains the second power of the unknown quantity and no higher power, is called a Quadratic Equation.

188. There are two cases.

I. When x^2 is present, but not x, as in $3x^2-27=0$, such an equation is called a *Pure* Quadratic Equation.

II. When x^2 is present, and also x, as in $3x^2-5x+2=0$, such an equation is called an *Adfected* Quadratic Equation, or simply a Quadratic Equation, the terms pure and adfected seldom coming into use.

189. The general principles and processes explained in Arts. 160–164 for the solution of simple equations apply also to quadratic equations, Axiom vi of Art. 160 coming into use for the first time, which in relation to Quadratics may run, The square roots (of like sign) of equals are equal.

190. For the present we have to do only with the main principle of the solution of quadratics. We therefore suppose all prior operations, which are necessary for the reduction of the equations to their simplest form, as integral expressions equated to zero, to have been performed.

Pure Quadratics.

191. To solve a Pure Quadratic, as $3x^2-27=0.$

Transpose, so that the known and unknown quantities may be on opposite sides. $3x^2=27.$

Secondly, divide by the coefficient of x^2, $x^2=9.$

Thirdly, take the square root of both sides, getting $x=\pm 3.$

192. It is our object in the case of an equation to find all possible values of x which will satisfy the equation. And both $+3$ and -3, when multiplied into themselves, yield $+9$. We must therefore give them both, which is done by the double sign.

Quadratic equations have all two *roots*, or values which satisfy them, whereas simple equations have but one. No greater generality would be gained by writing $\pm x=\pm 3$, and therefore the double sign is confined to the second number.

193. We do not assert that every square root of a number is equal to every square root of its equal, since this would involve

such a proposition as $+3=-3$. We have endeavoured to exclude this by saying in Axiom vi, *Like* roots of equals are equal. In the solution of an equation we do not assert that every value of x which offers a solution is equal to every other, but simply that each one (separately) satisfies the equation.

Exercises. No. 56.

(1) $5x^2-125=0$.

(2) $(x-5)(x-3)=16-8x$.

(3) $(2x-7)(3x-4)=244-29x$.

(4) $30(x^2-3)+6=6x^2+12$.

(5) $6x^2-\frac{3x^2}{4}-9=5x^2$.

(6) $\frac{2x^2-3}{3x^2-4}=\frac{4x^2-5}{6x^2-7}$.

(7) $\frac{x^2+7}{11}-\frac{2x^2-16}{3}+\frac{2x^2+5}{4}=5\frac{1}{3}+\frac{3x^2+7}{12}$.

(8) $\frac{5}{2x^2+1}=\frac{2}{5x^2-8}$.

(9) $\frac{3x^2-3}{4}-\frac{3x^2-4}{3}=5\frac{1}{3}-\frac{27+4x^2}{9}$.

Adfected Quadratics.

194. Every adfected quadratic can be immediately converted into a pure quadratic by the process called 'completing the square,' which we have already noticed in Art. 89.

We there saw that a trinomial of the form $x^2+2bx+c^2$ can be put in the form $x^2+2bx+b^2-(b^2-c^2)$,

or $(x+b)^2-(b^2-c^2)$.

Consequently $x^2+2bx+c^2=0$ becomes

$x^2+2bx+b^2=b^2-c^2$,

or $(x+b)^2=b^2-c^2$,

and, extracting the square root,

$$x+b=\pm\sqrt{b^2-c^2},$$

$$x=-b\pm\sqrt{b^2-c^2}. \quad \text{(i)}$$

195. Similarly, if we take the more general form of an expression $ax^2+bx+c=0$,

we may write it $x^2+\frac{b}{a}x=-\frac{c}{a}$,

$$\therefore\quad x^2+\frac{b}{a}x+(\frac{b}{2a})^2=(\frac{b}{2a})^2-\frac{c}{a}=\frac{b^2-4ac}{4a^2}.$$

Extracting the square root,

$$x+\frac{b}{2a}=\pm\frac{1}{2a}\sqrt{b^2-4ac};$$

$$\therefore\quad x=\frac{1}{2a}\{-b\pm\sqrt{b^2-4ac}\}. \quad \text{(ii)}$$

196. To complete the square in any particular case we have then this rule:—Having collected the terms containing unknown quantities in the first member, and transferred the other terms to the second member; divide both sides by the coefficient of x^2, and add to both sides the square of the half of the coefficient of x. Extract the square root, showing the double sign of the root on the second side, and transfer the known quantity which appears on the first side as the second term of the binomial root to the second side.

Example. $3x^2-5x+2=0.$

Transposing, $3x^2-5x=-2.$

Dividing, $x^2-\frac{5}{3}x=-\frac{2}{3}.$

Adding the square, $x^2-\frac{5}{3}x+(\frac{5}{6})^2=(\frac{5}{6})^2-\frac{2}{3},$

$$=\frac{25-24}{6^2}=\frac{1}{6^2}.$$

Extracting the root, $x-\frac{5}{6}=\pm\frac{1}{6},$

$$x=\frac{5\pm1}{6}=1 \text{ or } \frac{2}{3}.$$

197. The two roots of the equation are 1 and $\frac{2}{3}$, and it may perhaps be satisfactory to show that the two values thus obtained for x do equally satisfy the equation.

First, let $x = 1$, $\quad 3\times 1^2-5\times 1+2 = 3-5+2,$
$= 0.$

Next, let $x = \frac{2}{3}$, $\quad 3\times(\frac{2}{3})^2-5\times\frac{2}{3}+2 = \frac{4}{3}-\frac{10}{3}+2,$
$= 0.$

198. Instead of going through the process of completing the square in any particular case, we may make use of the general formula (ii)

$$x = \frac{1}{2a}\{-b\pm\sqrt{b^2-4ac}\},$$

adapting it to the particular instance by substituting the values of a, b, c.

Here $\quad a = 3, \quad b = -5, \quad c = 2;$

$$\therefore\ x = \frac{1}{6}\{5\pm\sqrt{25-24},\}$$

$$= \frac{1}{6}\{5\pm 1\},$$

$$= 1 \text{ or } \frac{2}{3}, \text{ as before.}$$

199. If $b^2 = 4ac$ the term $\sqrt{b^2-4ac} = 0$, and the two roots become equal.

This is the condition that the expression ax^2+bx+c may be a complete square.

200. If we take the form of a quadratic

$$x^2+2bx+c^2 = 0,$$

we learn from Art. 89 that

$$x^2+2bx+c^2 = \{x+b+\sqrt{b^2-c^2}\}\cdot\{x+b-\sqrt{b^2-c^2}\};$$

it appears that the expression is the product of two binomials, the last terms of which are the two roots with their signs changed. It follows that the coefficient of x with its sign changed is the sum of the roots, and the last term is their product. Hence, if we can discover by inspection the two factors of the expression on the left-hand side, we discover at the same time the roots, which are the numerical parts of the factors with their signs changed.

201. In some cases we shall find that the solution of a pure quadratic equation leads to results such as the following, $x^2 = 17$; where we are required to find the square root of a number which

is not a complete square. In this case we are satisfied to write $x = \pm\sqrt{17}$, without finding the approximate root (unless required to do so) by the arithmetical process. Such a quantity as $\sqrt{17}$ is called a Surd.

If the equation be expressed in general symbols we may arrive in like manner at a result, such as $x^2 = a$, giving $x = \pm\sqrt{a}$.

202. Again, the quadratic equation

$$x^2-4x+1 = 0, \quad \text{gives}$$
$$x^2-4x+4 = 3,$$
$$x = 2 \pm \sqrt{3},$$

and whenever the value of a, b, c in the quadratic

$$ax^2+bx+c = 0$$

is such that b^2-4ac is not a complete square, we have in like manner

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}, \text{ in the form}$$
$$x = a \pm \sqrt{\beta},$$

a quantity which contains a surd, or which is otherwise said to be partly rational and partly irrational.

Exercises. No. 57.

(1) $x^2-8x+15 = 0$.
(2) $x^2+12x+35 = 0$.
(3) $x^2-x-6 = 0$.
(4) $x^2+2x-35 = 0$.
(5) $x^2+2x-24 = 0$.
(6) $x^2-9x-22 = 0$.
(7) $6x^2-x-3 = 32$.
(8) $6x^2+15x = 10x+56$.
(9) $14x^2-37x+5 = 0$.
(10) $4x^2+4x+17 = 32$.
(11) $3x^2-7x = 6$.
(12) $4x^2-40x+99 = 0$.
(13) $\frac{x^2}{6}-\frac{x}{3} = \frac{x}{10}-\frac{1}{5}$.
(14) $\frac{x^2}{10}-\frac{3x}{5} = \frac{x}{2}-3$.
(15) $9\frac{3}{5}x-21\frac{15}{16} = x^2$.
(16) $9\frac{1}{3}x^2-90\frac{1}{3}x+195 = 0$.
(17) $\frac{18x^2}{5}+\frac{18078}{65}x+4728 = 0$.
(18) $\frac{2x^2}{3}-\frac{3x}{4}+\frac{5}{6} = \frac{11x^2}{12}-\frac{x}{2}+\frac{1}{3}$.

(19) $\frac{2x}{3}+\frac{x-1}{15}-\frac{22}{x}=\frac{2}{3}$.

(20) $\frac{x}{x+60}=\frac{7}{3x-5}$.

(21) $(x-1)(x-2)+2(x-2)(x-3)=0$.

(22) $x^3+2x-12=(x-2)(x-3)(x-4)$.

(23) $\frac{\frac{1}{2}(x-1)-1\frac{2}{3}}{2\frac{1}{2}+\frac{1}{3}(x+2)}=\frac{x+1}{33}$.

(24) $\frac{x^2-7}{x+3}=x-3+\frac{4-x}{x}$.

(25) $\frac{1}{x-2}-\frac{1}{x-3}+\frac{1}{x+1}=0$.

(26) $x+\frac{9}{x}=15(1-\frac{3}{x})$.

(27) $(x-1)(x-2)+2(x-2)(x-3)=70$.

(28) $(x-9)(x-10)+3(x-7)(x-8)=678$.

(29) $(x-2)(x-3)+(x-3)(x-4)=32$.

(30) $(x+1)(2x+3)=4x^2-22$.

(31) $\frac{x-a}{x+a}-\frac{x+a}{x-a}=\frac{3}{2}$.

(32) $(x-\frac{1}{2})(x-\frac{1}{3})+(x-\frac{1}{3})(x-\frac{1}{4})=(x-\frac{1}{4})(x-\frac{1}{5})$.

(33) $\frac{2x+3}{10-x}=\frac{2x}{25-3x}-6\frac{1}{2}$.

(34) $\frac{48}{x+3}+5=\frac{165}{x+10}$.

(35) $\frac{x-1}{x-2}+\frac{x-5}{x-6}=\frac{x-2}{x-3}+\frac{x-3}{x-6}$.

(36) $\frac{x-1}{x^2+x}-\frac{x+1}{x^2-x}=\frac{1}{4-2x}$.

(37) $\frac{x^2+a^2}{a-x}-\frac{x}{2}+\frac{3}{2}(x-\frac{1}{2}a)+\frac{15}{4}a=2a-x$.

(38) $(x-a)(x-b)=a(2a-b)$.

(39) $\frac{25x+180}{10x-81}=\frac{40x}{5x-8}-\frac{3}{5}$.

(40) $\frac{18+x}{6(3-x)} = \frac{20x+9}{19-8x} - \frac{13}{3-x}$.

(41) $\frac{x+7}{x+11} - \frac{x+5}{x+12} = \frac{47}{306}$.

(42) $\frac{x^2+b^2}{x^2-b^2} + \frac{2x+3b-a}{x+b} = \frac{2(x^2+bx-b^2)}{x^2-b^2}$.

(43) $2x(x+c) = (2a+c)x + ac$.

(44) $\frac{x^2+x+\frac{1}{2}}{a^2+1} + \frac{x^2+x}{a^2-1} = 0$.

(45) $\frac{x+4}{x-3} - \frac{2x-3}{x+4} = 7\frac{3}{8}$.

(46) $2\frac{a^3+x^3}{a+x} + \frac{a^3-x^3}{a-x} = 4a^2$.

(47) $(x^2+1)(x+2) = 2$.

(48) $abx^2 + \frac{3a^2x}{c} = \frac{6a^2+ab-2b^2}{c^2} - \frac{b^2x}{c}$.

(49) $\frac{2c^2}{d^2} + \frac{ac}{d} - (a-b)(2c+ad)\frac{x}{d} = (a+b)\frac{cx}{d} - (a^2-b^2)x^2$.

(6) Problems leading to Quadratics.

203. Any quadratic equation put into words, and addressed to any person as a demand, becomes a problem; and, inversely, many problems, expressed in algebraic language, take the form of quadratic equations.

Exercises. No. 58.

(1) Find a mean proportional between 3 and 27.

(2) What is that number, whose half, multiplied by its third, gives for product 864?

(3) What number is that whose square exceeds the number by 420?

(4) What number is that whose square exceeds ten times the excess of the number above 2 by 95?

(5) A rectangle whose length is 4 inches longer and breadth is 6 inches longer than the side of a certain square, contains twice as

many square inches as this square. Find the length of the side of the square.

(6) A horse-dealer buys a horse and sells it again for £144, and gains just as many pounds per cent. as the horse had cost him. What did he give for the horse?

(7) I thought of a number; I multiplied it by $2\frac{1}{3}$; I added 7 to the product; multiplied the result by 8 times the number thought of; I then divided by 14, and subtracted from the quotient four times the original number; the remainder was 2352: find the number thought of.

(8) Find a number such that its square exceeds it by 306.

(9) If a man can row from A to B with the stream in 45 minutes and takes 75 minutes in coming back, how long would he have taken if there had been no stream?

(10) A cistern can be filled by two pipes together in $1\frac{1}{3}$ hour. The larger pipe by itself will fill it sooner than the smaller by 2 hours. What time will each pipe separately take to fill it?

(11) Two numbers differ by 9 and their reciprocals by $\frac{1}{4}$; what are they?

(12) What number is that which exceeds its square root by 992?

(13) Two travellers, one walking a mile more in the hour than the other, start together to walk twenty miles; the first reaches his destination an hour before the other; what were the rates of walking?

(14) A number of square tiles were employed in paving two square courts, one of which had ten feet more in the side than the other: the courts having been arranged so as to form a single square, having ten feet more in the side than the larger of the former, it was found that the tiles just served to pave it. Find the size of each court.

(15) An express train started from London an hour later than a goods train, which it overtook and passed on another line of rails 30 miles from London: after 80 minutes longer they were 36 miles apart; find their rates of travelling.

(16) To divide the number 30 into two such parts, that their product shall be equal to 200.

(17) To divide the number $2m$ into two such parts, that their product shall be equal to m^2-n^2.

(18) A hollow square was formed of men, standing four deep; but having suffered a loss of one-eighth as many men as would have completed the solid square, it was re-formed, three deep, and with three fewer in each side: and then there were forty men to spare: how many were there at first?

(19) Divide the number 15 into two parts such that the sum of their cubes shall be 945.

(7) Simultaneous Equations of the Second Degree.

204. The object to be aimed at in solving simultaneous equations of the second degree is the same as in the treatment of such equations of the first degree: viz. to get rid of, or as it is called, *eliminate* one of the unknown quantities, when there remain an equation of one unknown quantity for solution. We can seldom however follow exactly the same order of proceeding.

Such equations as

$$\begin{cases} 2x^2+3y^2=35 \\ 3x^2+2y^2=30 \end{cases}$$

may be treated exactly on the same principles, for in fact x^2 and y^2 only come in the place of x and y.

We get $x^2=4$, $y^2=9$,

$x=\pm 2$, $y=\pm 3$;

and there are four pairs of values which satisfy the equation, viz.

$$\begin{cases} x=2, \\ y=3, \end{cases} \quad \begin{cases} x=2, \\ y=-3, \end{cases} \quad \begin{cases} x=-2, \\ y=3, \end{cases} \quad \begin{cases} x=-2, \\ y=-3. \end{cases}$$

205. A common form is the following:

$$\left.\begin{matrix} x+y=5 \\ xy=6 \end{matrix}\right\}, \quad \text{or} \quad \left.\begin{matrix} x-y=1 \\ xy=6 \end{matrix}\right\}.$$

In either of these cases we square the first equation and multiply the second by 4, add the result in the first case, subtract it in the second, then take the square root, proceeding as follows:—

$$\begin{array}{rl} x^2+2xy+y^2 = & 25 \\ 4xy = & 24 \\ \hline x^2-2xy+y^2 = & 1 \\ x-y = & \pm 1 \\ x+y = & 5 \\ \hline 2x = & 6 \text{ or } 4 \\ 2y = & 4 \text{ or } 6 \end{array} \qquad \begin{array}{rl} x^2-2xy+y^2 = & 1 \\ 4xy = & 24 \\ \hline x^2+2xy+y^2 = & 25 \\ x+y = & \pm 5 \\ x-y = & 1 \\ \hline 2x = & 6 \text{ or } -4 \\ 2y = & 4 \text{ or } -6 \end{array}$$

$$\begin{array}{l} x = 3 \text{ or } 2, \\ y = 2 \text{ or } 3; \end{array} \qquad \begin{array}{l} x = 3 \text{ or } -2, \\ y = 2 \text{ or } -3. \end{array}$$

The first pair of equations has the two pair of solutions,

$$\begin{cases} x = 3, \\ y = 2, \end{cases} \text{ and } \begin{cases} x = 2, \\ y = 3; \end{cases}$$

and the other has also two, viz.

$$\begin{cases} x = 3, \\ y = 2, \end{cases} \text{ and } \begin{cases} x = -2, \\ y = -3. \end{cases}$$

206. In equations of the form

$$\begin{cases} x^2-3y^2 = 6, \\ 2x-3y = 3, \end{cases}$$

where one of the equations contains terms of the second order and the other only terms of the first, one of the unknown quantities should be expressed in terms of the other from the second equation and then substituted in the first. Here

$$x = \tfrac{3}{2}(y+1);$$

$$\therefore \ \tfrac{9}{4}(y+1)^2-3y^2 = 6,$$

which is a quadratic. Each value of y determined from this equation must be substituted for y in the second, from which a value of x will result.

207. Sometimes an equation which is of a degree higher than the second may be reduced to the second degree by dividing by an expression the value of which is known from the second equation. For example—

$$x^3+y^3 = 9,$$
$$x+y = 3.$$

Dividing the first equation by the second, that is to say, the first member of the first equation by the first member of the second, and the second member of the first by the second member of the second, we have

$$x^2-xy+y^2 = 3.$$

Also $x^2+2xy+y^2 = 9$, by squaring $x+y$;

$\therefore\ 3xy = 6,\quad xy = 2;$

$\therefore\ x^2-2xy+y^2 = 1,$

$x-y = \pm 1,$

$x+y = 3,$

$x = 2 \text{ or } 1,$

$y = 1 \text{ or } 2.$

In a similar manner we might have proceeded if the equations had been

$$x^3+y^3 = 9,$$
$$x^2-xy+y^2 = 3.$$

208. Another case of frequent occurrence is such as

$$x^2+xy = 35,$$
$$xy-y^2 = 6,$$

where the left-hand members of the two equations are homogeneous and of the same dimensions. We may divide one equation by the other, member by member: thus,

$$\frac{x^2+xy}{xy-y^2} = \frac{35}{6}.$$

Hence $6x^2+6xy = 35xy-35y^2,$

$6x^2-29xy = -35y^2,$

$$x^2-\frac{29}{6}xy+\left(\frac{29}{12}\right)^2y^2 = \left(\frac{29}{12}\right)^2y^2-\frac{70}{12}y^2 = \frac{841-840}{144}y^2.$$

Taking the square root,

$$x-\frac{29}{12}y = \pm\frac{1}{12}y,$$

$$x = \frac{5}{2}y \text{ or } \frac{7}{3}y.$$

Substituting, first, one of these values of x in terms of y, and afterwards the other, in either of these original equations, we shall have a pure quadratic, giving

$$x = \pm 5, \qquad \pm\frac{7}{2}\sqrt{2};$$

$$y = \pm 2, \qquad \pm\frac{3}{2}\sqrt{2}.$$

The rule commonly given for such homogeneous equations is to assume a symbol, as v, to express the ratio $\frac{x}{y}$, so that $x = vy$. Then, dividing one equation by the other, member by member, we obtain a quadratic in v, which leads to the same result as before: In simple cases there is no necessity for the introduction of a new symbol for the ratio.

Exercises. No. 59.

(1) $x+y=28$, $xy=187$.

(2) $x+y=26$, $xy=169$.

(3) $x-y=8$, $xy=273$.

(4) $x-y=10$, $xy=299$.

(5) $x^2+y^2=650$, $xy=323$.

(6) $x^2+y^2=578$, $x+y=30$.

(7) $x^2+y^2=1258$, $x-y=4$.

(8) $x^2+y^2=613$, $x^2-y^2=35$.

(9) $x^2-y^2=248$, $x^2y^2=915849$.

(10) $x^2-y^2=315$, $xy=494$.

(11) $x^2-y^2=144$, $x+y=72$.

(12) $x^2-y^2=520$, $x-y=10$.

(13) $x^2+xy+y^2=607$, $xy=69$.

(14) $x^2+3xy+y^2=1741$, $xy=297$.

(15) $x^2-xy+y^2=1263$, $3xy=2706$.

(16) $x^2+xy+y^2=508$, $x+y=26$.

(17) $x^2+xy+y^2=777$, $x-y=6$.

(18) $x^3+y^3=7110$, $x+y=30$.

(19) $x^3-y^3=1647$, $x-y=3$.

(20) $x^2+xy+y^2=577$, $x^3-y^3=6347$.

(21) $x^2-xy+y^2=97$, $x^3+y^3=1843$.

(22) $4x^2+y^2=872$, $xy=182$.

(23) $4x^2+9y^2=985,$
$xy=72.$

(24) $4x^2+10xy+25y^2=5836,$
$xy=156.$

(25) $2x+3y=12,$
$5x^2-7y^2=17.$

(26) $3x-5y=24,$
$2x^2-30y^2=68.$

209. No further remark will be necessary in reference to problems producing simultaneous equations of the second order.

Exercises. No. 60.

(1) The sum of two numbers is 12, and their product is 20: find them.

(2) The sum of two numbers is 31, and their product is 228: find them.

(3) The sum of the squares of two numbers is 125, and their sum is 13: find them.

(4) The sum of the squares of two numbers is 410, and their sum is 28: find them.

(5) The difference of two numbers is 2, and the sum of their squares 394: find them.

(6) To find two numbers the product of which is 750 and the ratio $3\frac{1}{3}$.

(7) To find two numbers such that the sum of their squares is 13001, and the difference of the same 1449.

(8) To find two numbers in the ratio of 3 to 4, such that the sum of their squares is 324900.

(9) To find two numbers in the ratio of m to n, such that the sum of their squares is b.

(10) To find two numbers such that their sum is 9, and the sum of their cubes 243.

(11) To find two numbers such that their difference is 2 and the difference of their cubes 602.

(12) To divide the number 16 into two parts, such that if to their product we add the sum of their squares, the result shall be 208.

(13) To divide the number 39 into two parts, such that the sum of their cubes shall be equal to 17199.

(14) Find two numbers such that their sum may be 20, and the mean proportional between them 8.

CHAPTER X.

INVESTIGATION OF METHODS.

(1) Involution and Evolution.

210. The third application of Algebra is to the investigation of new methods and theorems, and particularly to the discovery of inverse methods. The mutually inverse methods of Involution and Evolution will furnish our illustration.

Involution is the general name for the multiplication of an expression into itself, so as to form its second, third, fourth, or any other power. We are already familiar with the simple cases of the square and cube of a binomial. Thus, we know that

$$(a+b)^2 = a^2 + 2ab + b^2,$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

And here we may see what makes Algebra of value for our present purpose, viz. that whereas in Arithmetic a number, such as 5329, bears no marks of the processes by which it was formed, we trace in the algebraical expressions the original parts of the expression, and also the law of formation of the result.

For the first term is the first part a of the binomial raised to the same power as the binomial; the second term has the index of the binomial for coefficient, whilst in each successive term the index of a is diminished, and that for b increased by 1. Such observations lead, as we shall see later, to a general method for

writing down any power of a binomial, called the Binomial Theorem.

211. Evolution is the process inverse to Involution, and answers the question, What expression is that which, being raised to a given power, produces a given expression? The simplest cases are the methods of finding the Square Root and the Cube Root of an expression.

(2) Square Root.

212. We assume that the square root and cube root of a single term can be found; thus, $\sqrt{a^2} = a$, $\sqrt[3]{b^3} = b$. It should be noticed, however, that since

$$(+a)\times(+a) = +a^2, \quad \text{and} \quad (-a)\times(-a) = +a^2,$$

$\sqrt{a^2}$ has strictly two values, and it will be found hereafter that there are, in a sense to be then explained, three cube roots, four fourth roots, and so on. But in all these cases there is one principal root, which we shall call the *primary* root, from which the root may be derived. This answers to the arithmetical root of a number, and it is this root which we seek in the present investigations. In square root the root in question is the positive root, from which the negative root is derived by taking the same value with the negative sign.

213. By actual multiplication,

$$(a+b)^2 = a^2 + 2ab + b^2.$$

Inversely, $\sqrt{a^2 + 2ab + b^2} = a + b$.

We observe that if the quadratic expression and the simple expression, or root, be similarly arranged according to powers of the same letter,

(i) The first term of the root, a, is the square root of the first term of the quadratic expression.

Hence the rule:—Having arranged the expression by the powers of some letter, place the square root of the first term in the quotient's place, and its square under the first term of the expression; subtract and bring down the remainder, as in Example 1 below.

(ii) The second term, b, of the root may be derived from the first term of the remainder, $2ab$, by dividing it by $2a$.

Hence the rule:—Write the double of the root already found as a divisor to the remainder, leaving space for another term. Dividing by it write the quotient as a second term of the root in the quotient's place. (See Example 1.)

(iii) The full remainder $= 2ab + b^2 = (2a + b) \times b$.

Hence the rule:—Write the new quotient as a second term in the divisor's place; multiply the divisor thus formed by the quotient and subtract the product from the remainder. If there be no remainder, as supposed above, the exact root is found; should there be other terms, the complete square $(a+b)^2$, which has been withdrawn at two subtractions, answers to a^2, in the expression from which we started, and the same process may be continued as before. (See Example 1.)

Example 1.

$$\begin{array}{rl} a^2 + 2ab + b^2 & \big(a + b \\ a^2 & \\ \hline 2a + b \big) \; 2ab + b^2 & \\ \quad 2ab + b^2 & \\ \hline \quad * \quad * & \end{array}$$

Example 2.

$$\begin{array}{lll} 1 & 0 & a^2 + 2ab + b^2 \big(a + b \\ & a & a^2 \\ & \overline{2a} & \\ & 2a + b & \quad 2ab + b^2 \\ & \overline{2a + 2b} & \quad 2ab + b^2 \\ & & \quad * \quad * \end{array}$$

214. Any process by which we can form the divisors $2a$ and $2a+b$, and from this last the product $(2a+b)b$, will serve the purpose. We recommend the use of Horner's process, already partially explained in Art. 98, since it applies to all cases of

evolution, arithmetical and algebraical, and to the numerical solution of equations of all orders.

Arrange the expression as before, placing before it in the same line, at convenient intervals, as in Example **2**, the figure 1, called the initial unit, and a cipher. Find the square root of the first term, a; place it in the quotient's place, and employ it as a multiplier, thus:—Beginning at the initial unit, multiply it by a, carry the product and add it to the next column. Multiply this result again by the same multiplier, and carry the product, a^2, to the last column. *In this last column subtract instead of adding* (the object being now to withdraw the square a^2), and bring down the remainder. Repeat the process a second time with a, stopping short one column sooner. That is to say, multiply 1 by a, and add the result a to a already found in the second column, giving $2a$. The work is now complete as regards the quotient a. If we call the work once along the line Horner's single process, we may call this Horner's complete process. We have given the rule in detail, because we have nothing further to do than to repeat it continually, with any new quotients.

For the next quotient, divide the first term of the remainder by the expression (or rather its first term) in the previous column; this gives the quotient b. Place it in the quotient's place and work with it as before with a. The second result $2a+2b$ in the second column is only wanted if there should be further operations to follow.

Example 3.

$$
\begin{array}{llll}
1 & 0 & a^2r^4+2abr^3+(b^2+2ac)r^2+2bcr+c^2 & \big(ar^2+br+c \\
 & ar^2 & a^2r^4 & \\
 & \overline{} & & \\
 & 2ar^2 & 2abr^3+(b^2+2ac)r^2 & \\
 & 2ar^2+br & 2abr^3+b^2r^2 & \\
 & & \overline{} & \\
 & 2ar^2+2br & & 2acr^2+2bcr+c^2 \\
 & 2ar^2+2br+c & & 2acr^2+2bcr+c^2 \\
 & & & \overline{} \\
 & & & * \quad * \quad * \\
 & & & \overline{\overline{}}
\end{array}
$$

Example 4.

$$
\begin{array}{llll}
1 & 0 & 1+4x+10x^2+12x^3+9x^4 & (1+2x+3x^2 \\
 & 1 & 1 & \\
 & \hline & & \\
 & 2 & 4x+10x^2 & \\
 & 2+2x \cdot & 4x+4x^2 & \\
 & \hline & & \\
 & 2+4x & 6x^2+12x^3+9x^4 & \\
 & 2+4x+3x^2 & 6x^2+12x^3+9x^4 & \\
 & & \hline & \\
 & & * \quad * \quad * & \\
 & & \hline\hline &
\end{array}
$$

The process may be somewhat compressed in the second column, by placing the result from the new quotient in the same line as the previous result. But we have preferred for the sake of the beginner to show the work in detail.

The chief points to be remembered are;—always to begin each line of work from the initial unit and to repeat the work in the second column *a second time for every quotient.*

We now give a case in which the root takes the form of an infinite series.

Example 5. Find $\sqrt{1+x}$.

$$
\begin{array}{llll}
1 & 0 & 1+x & \left(1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\&c.\right. \\
 & 1 & 1 & \\
 & \hline & & \\
 & 2 & x & \\
 & 2+\frac{x}{2} & x+\frac{x^2}{4} & \\
 & \hline & & \\
 & 2+x & -\frac{x^2}{4} & \\
 & 2+x-\frac{x^2}{8} & -\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{64} & \\
 & & \hline & \\
 & & \frac{x^3}{8}-\frac{x^4}{64} & \\
 & & \hline\hline &
\end{array}
$$

215. The method of finding the square root of a number is taught in Arithmetic. It is however desirable that the relation of the arithmetical rule to the above investigation should be pointed out.

A number of two digits may be represented by the form $a \cdot 10 + b$, of three digits by $a \cdot 10^2 + b \cdot 10 + c$, and so on, a number of n digits being represented by

$$N = a \cdot 10^{n-1} + b \cdot 10^{n-2} + c \cdot 10^{n-1} + \ldots + p \cdot 10 + q,$$

where $b, c, \ldots p, q$ may have any values from 0 to 9, and a any value from 1 to 9.

Thus N ranges from 10^{n-1} (inclusive) to 10^n (exclusive), the last being *the first number* which has $(n+1)$ figures. Therefore N^2 ranges from 10^{2n-2} (inclusive) to 10^{2n} (exclusive), i. e. it comprises all numbers which have $2n-1$ or $2n$ figures. Hence the rule for pointing:—Mark off pairs of figures, beginning from the right-hand. The number of periods marked off will show the number of figures in the root.

Again, $N^2 = a^2 \cdot 10^{2n-2} + 2ab \cdot 10^{2n-3} + \ldots$;

but it does not follow that a^2 represents the period on the left-hand or the leading period, since $2ab$ may be > 10. Still,

$$N < (a+1) \cdot 10^{n-1}, \text{ even if } b, c, \ldots \text{ are all 9's};$$

$$\therefore \quad N^2 < (a+1)^2 10^{2n-2}.$$

(i) Thus the square root of the *greatest square* contained in the first period is a.

(ii) $b \cdot 10^{n-2}$ is found by using $2a \cdot 10^{n-1}$ as a trial divisor, as before. This trial, however, no longer gives the quotient exactly, but only approximately, and regard must be had to the condition that the *full remainder* formed with it cannot exceed the actual remainder. The process is continued as before. In practice, unnecessary ciphers are dispensed with by keeping the figures under their own grades and bringing down a period at a time, as wanted. The details may be seen in *Scholar's Arithmetic*, Chap. XV.

Exercises. No. 61.

Find the square roots of

(1) $9x^2 - 12xy + 4y^2$.

(2) $x^2 + x + \frac{1}{4}$.

(3) $\frac{9x^8}{4} - 2x^4y^2 + \frac{4y^6}{9}$.

(4) $\frac{25}{4}a^2b^2 - \frac{5}{3}abc^2 + \frac{1}{9}c^4$.

(5) $9x^2-30ax-3a^2x+25a^2+5a^3+\frac{a^4}{4}$.

(6) $4a^2-12ab+9b^2+20ac-30bc+25c^2$.

(7) $9a^2-6ab+30ac+6ad+b^2-10bc-2bd+25c^2+10cd+d^2$.

(8) $9x^2-12+\frac{4}{x^2}$.

(9) $\frac{25a^2}{x^2}-20+\frac{4x^2}{a^2}$.

(10) $x^4-4x^2+12-\frac{16}{x^2}+\frac{16}{x^4}$.

(11) $9y^2-30y+31-\frac{10}{y}+\frac{1}{y^2}$.

(12) $a^2-6ab+9b^2-22ac+66bc+121c^2$.

(13) $x^2+4-\frac{6}{x}+\frac{4}{x^2}-\frac{12}{x^3}+\frac{9}{x^4}$.

(14) $1-4x+10x^2-20x^3+25x^4-24x^5+16x^6$.

Find to 4 terms:—

(15) $\sqrt{1-x^2}$. (16) $\sqrt{1-x-x^2}$.

(17) $\sqrt{1-x+x^2-x^4+\ldots}$

216. We now come to Cube Root.

Since $(a+b)^3=a^3+3a^2b+3ab^2+b^3$, we may obtain a at once as the cube root of the first term; and we may find b from the second term by using $3a^2$ as a divisor: the next operation is to form the full divisor $3a^2+3ab+b^2$, which multiplied by b gives the full remainder.

We shall show that Horner's process does this, and it is the most convenient method.

Arrange the cubic expression according to powers of some letter, and before it in the same line an initial unit and two ciphers at convenient intervals: find the cube root a and place it in the quotient's place; then employ it as a multiplier in Horner's single process along the line, giving in the successive columns a, a^2, a^3. Subtract this last result and bring down the remainder. Repeat the process from the initial unit, but stopping one column short; and repeat it once more, still stopping one column short of the

last result. This is the complete process for a. Use $3a^2$ as the divisor of the first term of the remainder, viz. $3a^2b$, to find b. Write b as a second term of the root in the quotient's place, and work with b exactly as before with a, and so on, until there is no remainder; or until a sufficient number of terms have been found.

The arrangement is shown below.

Example 1.

1 0	0	$a^3+3a^2b+3ab^2+b^3$	$(a+b$
a	a^2	a^3	
$2a$	$3a^2$	$3a^2b+3ab^2+b^3$	
$3a$	$3a^2+3ab+b^2$	$3a^2b+3ab^2+b^3$	
$3a+b$	$3a^2+6ab+3b^2$	* * *	
$3a+2b$			
$3a+3b$			

It will be seen that $3a^2$ and $3a^2+3ab+b^2$ are formed successively in the second column, as required. The process has been carried further than was necessary, so as to be ready for a further operation, if required.

Example 2. Find $\sqrt[3]{1+x}$.

1 0	0	$1+x$	$(1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3$
1	1	1	
2	3	x	
3	$3+x+\frac{1}{9}x^2$	$x+\frac{1}{3}x^2+\frac{1}{27}x^3$	
$3+\frac{1}{3}x$	$3+2x+\frac{1}{3}x^2$	$-\frac{1}{9}x^2-\frac{1}{27}x^3$	
$3+\frac{2}{3}x$	$3+2x-\frac{1}{9}x^3+\frac{1}{81}x^4$	$-\frac{1}{9}x^2-\frac{2}{9}x^3+\frac{1}{81}x^5-\frac{1}{729}x^6$	
$3+x$			
$3+x-\frac{1}{9}x^2$		$\frac{5}{27}x^3-\frac{1}{81}x^5+\frac{1}{729}x^6$	

217. The adaptation to Arithmetic is similar to that in Square Root.

N^3 ranges from 10^{3n-3} (inclusive) to 10^{3n} (exclusive), i.e. it comprises all numbers which have $3n-2$, $3n-1$, or $3n$ figures, and by pointing off periods of 3 figures each from the right-hand we shall have the number of figures in the root.

Also, if the leading digit be a, the lowest value of the first period is a^3, and it cannot amount to $(a+1)^3$, since

$N < (a+1)10^{n-1}$, and therefore $N^3 < (a+1)^3 10^{3n-3}$.

Hence a may be found and $3a^2+3ab+b^2$ formed, as above, by Horner's process.

Exercises. No. 62.

Find the cube roots of:—

(1) $8x^3-60x^2+150x-125$.

(2) $8a^3-84a^2x+294ax^2-343x^3$.

(3) $y^3-6y+\frac{12}{y}-\frac{8}{y^3}$.

(4) $27z^6-54az^5+63a^2z^4-44a^3z^3+21a^4z^2-6a^5z+a^6$.

(5) $8y^3-36y^2+54y+60y^2z-27-180yz+135z+150yz^2-225z^2+125z^3$.

Find to four terms :—

(6) $\sqrt[3]{1-2x^2}$.

(7) $\sqrt[3]{1-x-x^2}$.

(8) $\sqrt[3]{a^3-x^3}$.

(9) $\sqrt[3]{1-3x+2x^2}$.

Find the cube roots of

(10) $125z^3-525az^2+735a^2z-343a^3$.

(11) $1+9x+18x^2-27x^3-54x^4+81x^5-27x^6$.

(12) $x^3-3x^2+9x-13+\frac{18}{x}-\frac{12}{x^2}+\frac{8}{x^3}$.

(13) $27a^3-54a^2b+36ab^2+54a^2c-8b^3-72abc+24b^2c+36ac^2-24bc^2+8c^3$.

(14) $1-9x+27x^2-27x^3+6y-36xy+54x^2y+12y^2-36xy^2+8y^3$.

Remark.—The same process applies to the fourth, fifth, or any root; the number of columns, exclusive of the initial unit, and also the number of repetitions, stopping a column short each time, corresponding to the number of the root.

CHAPTER XI.

SUPPLEMENTARY.

This place is chosen for a few supplementary remarks and additions.

(1) Unknown Quantities.

218. We have already remarked that unknown quantities need not necessarily be denoted by the later letters of the alphabet, x, y, z, this being simply a matter of convenience. Any formula, giving a result in terms of certain data, may be employed to find, inversely, the values of one (or more) of the data, which will lead to a given result. The symbols representing these data then become unknown quantities, whatever letters may have been employed.

219. For example, if we take the formula for the sum of an A. P., viz. $s = \frac{1}{2}n\{2a+(n-1)d\}$, n is an unknown quantity, if a, d, and s be given and the number of terms be demanded.

Example. $a = 6$, $d = 3$, $s = 42$; required n.

Here
$$\tfrac{1}{2}n\{12+3(n-1)\} = 42,$$
$$3n^2+9n = 84,$$
$$n^2+3n = 28, \text{ a quadratic;}$$

which gives
$$n = 4 \text{ or } n = -7.$$

The first of these values gives a direct answer to the demand, the four terms being 6, 9, 12, 15: the other value, $n = -7$, does not directly answer the demand, since a negative number of terms is inconceivable. Here however, as often happens, there is an allied problem to which the solution directly belongs. The formula

$$s = \tfrac{1}{2}n\{2a+(n-1)d\}$$

may be arranged

$$s = \tfrac{1}{2}(-n)\{-2a-nd+d\},$$
$$= \tfrac{1}{2}(-n)\{2(d-a)+(-n-1)d\};$$

which is a similar form to the other, $-n$ coming in the place of n, and $d-a$ in the place of a. Thus it appears the negative value of

n answers to the number of terms of a series, the first term of which is $d-a$, and the sum s. In the example given above we have the series $-3, 0, 3, 6, 9, 12, 15$, the sum of which is 42, as before.

The second value of n is often fractional, and in this case no exact number of terms is indicated by it as a solution, but we learn that a fewer number of terms would be in defect, and a larger number in excess, of the assigned sum.

(2) Inequalities.

220. An inequality is an assertion that the relation of equality does not exist between two quantities.

The sign $\neq$ is sometimes used to signify 'is not equal to,' as $\frac{a}{b}+\frac{b}{a}>2$, provided $a \neq b$. But, much more frequently, the signs $>$ and $<$ are employed in the statement of inequalities. We have already had occasion to notice one or two inequalities: but we desire to direct attention to their general relation to equalities. In general, the methods of transforming equalities from one form to another, step by step, which we have exhibited in Chapter IX, are applicable to inequalities, all the axioms in Art. 160 having corresponding forms for inequalities, as

i. If equals be added to unequals, the results are unequal.

ii. If equals be multiplied by unequals, the results are unequal, and so on: and the same signs, whether $>$ or $<$, continue to apply. Remark, however, the following differences in applying the operations I, II, &c. of Art. 162. If the two sides of an inequality be interchanged, the sign $>$ must be replaced by $<$ and vice versâ. Thus, $a>b$ implies $b<a$. So also if the members of an inequality be subtracted from equals; thus, $a>b$ implies $c-a<c-b$. As one case of this, $a>b$ implies $-a<-b$, algebraically. The same must be done if equals be divided by the members of an inequality. Thus, $a>b$ implies $\frac{c}{a}<\frac{c}{b}$, and as one case, $a>b$ implies $\frac{1}{a}<\frac{1}{b}$.

221. The applications of Axioms v, iii, ii are seen in the following demonstration:—The arithmetic mean of two quantities is greater than the geometric mean.

Let a and b be the two quantities: then the arithmetic mean is $\frac{1}{2}(a+b)$, and the geometric mean is $\sqrt{ab}$. And

$$\begin{aligned} &\tfrac{1}{2}(a+b) > \sqrt{ab}, \quad \text{if} \\ &\tfrac{1}{4}(a+b)^2 > ab, \quad \text{if} \\ &a^2+2ab+b^2 > 4ab, \quad \text{if} \\ &a^2-2ab+b^2 > 0, \end{aligned}$$

which is true, since a and b are different quantities.

Example.—The arithmetic mean of 1 and 9 is 5, whilst the geometric mean is 3.

(3) Fractional and Negative Indices.

222. It is now time to mention a very convenient extension of the index notation which is extremely characteristic of the method of interpreting symbols in Algebra.

The first definition of an index, Art. 37, that it represents the number of like factors in a product, evidently supposes that the product is a whole number, and the general properties

$$a^m \times a^n = a^{m+n} \quad \text{(i)}$$

$$a^m \div a^n = a^{m-n} \quad \text{(ii)}$$

and

$$(a^m)^n = a^{mn} \quad \text{(iii)}$$

were obtained in Arts. 38 and 39 on the supposition that m and n were integers. But in Algebra secondary properties of this kind attain such an importance and are employed with such freedom that it is inconvenient to have their universality restricted by any conditions of this kind, and the question arises,—Can any interpretation be put upon those cases, which seem anomalous, and which were not contemplated in the original definition, which will be consistent with the properties now under consideration? It will be seen that this can be done.

223. Since $a^{m-n} = a^m \div a^n$, if we put $m = n+1$, we shall have

$$a^1 = a^{n+1} \div a^n = a.$$

Again, if in the same equality we put $m = n$, we have

$$a^{n-n} = a^n \div a^n, \quad \text{or} \quad a^0 = 1.$$

224. Again, if in $a^{m-n} = a^m \div a^n$ we put $m = 0$, which we may now do, in accordance with the interpretation just given to a^0, we have

$$a^{-n} = a^0 \div a^n,$$
$$= 1 \div a^n,$$
$$= \frac{1}{a^n}.$$

Example. $a^{-1} = \frac{1}{a}$, $a^{-2} = \frac{1}{a^2}$, $a^{-3} = \frac{1}{a^3}$, $a^{-6} = \frac{1}{a^6}$.

225. Again, if we suppose that m may be a fractional quantity $= \frac{1}{n}$, we shall have, in accordance with (iii), $(a^{\frac{1}{n}})^n = a^1$, since mn then $= 1$; therefore $(a^{\frac{1}{n}})^n = a$, and, taking the n^{th} root of both sides, $a^{\frac{1}{n}} = \sqrt[n]{a}$.

Example. $a^{\frac{1}{2}} = \sqrt{a}$, $a^{\frac{1}{3}} = \sqrt[3]{a}$,

and $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$,

$a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^1 = a$.

So also, putting m in (iii) $= \frac{p}{q}$, and $n = q$, we have

$$(a^{\frac{p}{q}})^q = a^{\frac{p}{q} \times q} = a^p;$$

and, taking the q^{th} root of both sides,

$$a^{\frac{p}{q}} = \sqrt[q]{a^p}.$$

Also $$a^{\frac{p}{q}} = (a^{\frac{1}{q}})^p = \{\sqrt[q]{a}\}^p.$$

Example. $a^{\frac{2}{3}} = \sqrt[3]{a^2} = (\sqrt[3]{a})^2$.

Questions and Exercises. No. 63.

Find the number of terms in the following series in A. P.:—

(1) $a = 9$, $d = 5$, $s = 87$.

(2) $a = -5$, $d = 2$, $s = 7$.

(3) $a = -4$, $d = 3$, $s = 72$.

(4) $a = -4$, $d = \frac{3}{2}$, $s = \frac{7}{2}$.

(5) Prove that $\frac{1}{3}(a+b+c) > \sqrt[3]{abc}$, unless a, b, c are all equal.

(6) State in words the process of solving a quadratic equation, consisting of a trinomial in its simplest form equated to zero.

(7) State in words the process of solving two simultaneous equations, one of which gives the value of the sum of two unknown quantities, and the other the value of their product.

(8) State the process, when the sum of the squares is given, and also the product of the unknown quantities.

Find the value of the following :—.

(9) $(a^{\frac{1}{2}}b^{\frac{1}{2}})^4$ and of $(a^{\frac{1}{2}}+3b^{\frac{1}{2}})^2$.

(10) $(a^{\frac{1}{3}}b^{\frac{1}{3}})^4$ and of $(a^{\frac{1}{3}}b^{\frac{2}{3}})^6$.

(11) $(a^{\frac{2}{3}}-3a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}})^2$.

(12) $(2a^{\frac{1}{3}}-3b^{\frac{1}{3}})^3$.

Find the first four terms of the following, using the fractional index notation :—

(13) $\sqrt{(a-2x)}$.

(14) $(a-2x)^{-\frac{1}{3}}$.

(15) $(1-3x-5x^2)^{-\frac{1}{2}}$.

(16) $(2-5x+7x^2)^{\frac{1}{3}}$.

(17) Find the condition that a quadratic equation may have equal roots.

Multiply out the following :—

(18) $(2x^{\frac{1}{2}}-3x^{\frac{1}{3}})(3x^{\frac{1}{2}}+5x^{\frac{1}{3}})$.

(19) $(3x^{-\frac{1}{2}}+2x^{-\frac{1}{3}}+5x^{-\frac{1}{4}})(2x^{\frac{3}{2}}+3x^{\frac{2}{3}}+x^{\frac{2}{3}})$.

(20) $(2x^{-1}+4x^{-2}+x^{-3})(x^{-2}+2x^{-4}-3x^{-5})$.

(21) $(x^{\frac{p}{q}}-x^{\frac{r}{s}})(x^{\frac{m}{n}}+x^{\frac{t}{u}})$.

(22) Find $(x^{\frac{p}{q}}-x^{\frac{r}{s}})^3$.

PART III.

APPLICATIONS OF ALGEBRA (*continued*).

226. IN Part III we further explain and illustrate the applications of Algebra (i) to the expression of general results in formulae, (ii) to the discovery of unknown numbers, (iii) to the investigation of methods, as in Part II.

CHAPTER XII.

ON ALGEBRAICAL FORMULAE.

(1) Permutations and Combinations.

227. If we have n letters and form out of them all the possible groups of r letters each, irrespective of the order in which they are arranged, these are called their combinations. Thus the combinations of three letters[1] a, b, c, two together, are ab, bc, ca.

Combinations are turned into *permutations*, when we distinguish the orders in which the letters stand.

We may have either permutations in which the letters are taken r together ($r < n$), or in which they are taken n together, all entering into every group. These last are usually called simply Permutations. They are supposed to be taken all together, unless it is otherwise specified.

---

[1] The letters are not here symbols of numbers: they stand as designations of things or persons.

The permutations of *a*, *b*, *c*, taken two together, are *ab*, *ba*, *ac*, *ca*, *bc*, *cb*.

The permutations of *a*, *b*, *c*, taken all together, are *abc*, *acb*, *bca*, *bac*, *cab*, *cba*.

Permutations of n Things.

228. To find a formula for the number of permutations of n things.

If we had one thing, *a*, alone, it could be placed only in one way; we should have only 1 permutation. Two things, *a*, *b*, give the permutations *ab*, *ba*, 2 in number. If there are three things, *a*, *b*, *c*, we may place *a* first, and then the other letters may be arranged in two ways, *bc* or *cb*. Similarly, we may have two ways in which *b* stands first, and two in which *c* stands first, that is, 2×3 or 6 permutations in all.

The permutations of four things, *a*, *b*, *c*, *d*, are those in which *a* stands first (and this is the same as the number of permutations of the three things remaining, or 1.2.3), and then those in which *b*, *c*, *d* stand first. The full number is 1.2.3.4, or 24. The manner in which we pass from two things to three, by introducing the factor 3, from three things to four, by introducing the factor 4, shows us clearly that, continuing in the same way, we shall introduce at every step a new higher factor, corresponding to the number of things to be permuted, so that the number of permutations of n things $= 1.2.3 \ldots n$. This expression is written for brevity $\lfloor\underline{n}$, and is called 'Factorial n.'

Example.—The number of permutations of 5 things $= 1.2.3.4.5$, or 120.

229. The above is a good illustration of one of the methods of obtaining a general formula, viz. (i) by Generalization. Other methods are (ii) by Induction, (iii) by Chain Rule.

We see that at each step a new number enters in a manner, the law of which is so obvious, that there is no risk in passing from a few instances to the general law.

230. It is only in very simple cases that this is accepted as

sufficient, and it is a universal rule, that the correctness of every generalization should be verified by giving to the general symbol particular values, and comparing the results deduced from the formula with those obtained by trial in these particular cases.

Permutations taken r together.

231. To find the number of permutations of n things taken r together.

Let the n things be $a, b, c, d, \ldots$.

Suppose a placed before each of them, except itself; this will give $n-1$ permutations taken 2 together, in which a stands first. The same for each of the n letters, when standing first. There are therefore in all $n(n-1)$ permutations taken 2 together.

Next, suppose a to be placed before all the permutations, taken 2 together, of the remaining $n-1$ letters, $b, c, d,$ &c.

To discover the number of these we must change n in the last formula into $n-1$, giving $(n-1)(n-2)$. This is therefore the number of permutations taken 3 together, in which a stands first, and as there will be just as many for each of the n letters, standing first, we have, on the whole, $n(n-1)(n-2)$ permutations taken 3 together. These two cases suggest the law, That the number of permutations of n things taken r together will be the product of r consecutive numbers descending from n, which we may call a descending factorial from n of r factors;

$$= n(n-1)(n-2)\ldots(n-r+1).$$

232. This is, like the last, a simple generalization, and might be accepted, as sufficiently proved, after being verified in particular instances. But a suspicion is supposed to rest upon generalizations, which are not of extreme simplicity, as if they were dictated by a previous acquaintance with the result. To avoid this, the formula is treated as hypothetical, to be otherwise rigidly established, as follows:

233. Suppose that ${}^{r}P_{n}$ signifies the number of permutations of n things taken r together, and let us assume that the hypothetical formula is true for n things taken $r-1$ together, n being quite

general, but r for the present limited to a particular value. Accordingly,

$$^{r-1}P_n = n(n-1)\dots(n-r+2),$$

and $$^{r-1}P_{n-1} = (n-1)(n-2)\dots(n-r+1).$$

If now we have n things, a, b, c, ... and we leave a out for a moment, and if we then form all the permutations of the remaining $n-1$ things, taken $r-1$ together, and, finally, prefix a to each, we shall have,

$^{r-1}P_{n-1}$ permutations in which a stands first,

$^{r-1}P_{n-1}$ permutations in which b stands first,

and so on through all the n things: therefore, on the whole,

$$^rP_n = n \times {}^{r-1}P_{n-1}, \qquad \text{(i)}$$

or $$^rP_n = n(n-1)(n-2)\dots(n-r+1), \quad \text{(ii)}.$$

Thus, if the formula (ii) holds for any one integral value of r, it holds for the next higher integer. But we saw in the last Article that it holds for $r = 2$, and for $r = 3$: hence it holds for $r = 4$, and hence for $r = 5$, and so on, step by step, for all integral values, both of n and r.

234. This method of proof is called Mathematical Induction. It must not be confounded with the process of thought by which in science we rise from particular instances to a general law.

Mathematical Induction is a *demonstration* of the truth of a formula, *conditionally* on its holding for a lower value of some symbol, enabling us from the proof of its truth in some particular instances to establish its universality. The method is of wide application in Mathematics.

235. Another method, which is frequently convenient in regard to formulae which involve only a single term, is analogous to Chain Rule in Arithmetic.

Chain Rule supposes that we have a succession of equations, such that a factor of the right-hand member of any one equation appears in the left-hand member of the next. Multiplying all the right-hand members together, and also all the left-hand, all these factors, except the first and the last cancel, give at once a relation between A and L. (See *Scholar's Arithmetic*, Chapter XIV.)

Thus, if

$$aA = mB,$$
$$bB = nC,$$
$$cC = pD,$$
$$\vdots$$
$$kK = sL,$$
$$A = \frac{mnp \dots s}{abc \dots k} L.$$

236. To apply this.

As before, leaving out a from the n things $a, b, c, \dots$ let ${}^{r-1}P_{n-1}$ denote the number of permutations of the remaining $n-1$ things $b, c, \dots$. Placing a before each of them, ${}^{r-1}P_{n-1}$ is the number of permutations of n things taken r together in which a stands first. There will be just as many for $b, c, \dots$ standing first: hence

$$ {}^{r}P_{n} = n \times {}^{r-1}P_{n-1}, \quad \text{(i)}$$

and this is perfectly general: we may therefore replace n by $n-1$ and r by $r-1$, again and again,

$$\therefore \quad {}^{r-1}P_{n-1} = (n-1) \times {}^{r-2}P_{n-2},$$
$$ {}^{r-2}P_{n-2} = (n-2) \times {}^{r-3}P_{n-3},$$
$$\vdots$$

down to $$ {}^{2}P_{n-r+2} = (n-r+2) \times {}^{1}P_{n-r+1};$$

for we cannot have permutations less than 1 together: and, since the suffix always exceeds the prefix by $n-r$, its corresponding value is $n-r+1$. Also ${}^{1}P_{n-r+1} = n-r+1$.

Hence, by Chain Rule,

$$ {}^{r}P_{n} = n(n-1)(n-2)\dots(n-r+2) \, {}^{1}P_{n-r+1},$$
$$= n(n-1)(n-2)\dots(n-r+2)(n-r+1). \quad \text{(ii)}$$

Example.—The number of permutations of 5 things taken 3 together $= 5.4.3 = 60$.

237. The formula (ii) may also be written

$$ {}^{r}P_{n} = \frac{\lfloor n}{\lfloor n-r}; \quad \text{(iii)}$$

or, again, if we put $n-r = m$, $n = m+r$;

$$ {}^{r}P_{m+r} = \frac{\lfloor m+r}{\lfloor m}. \quad \text{(iv)}$$

238. If we put $r = n$, and use P_n as the symbol for the number of permutations of n things taken n together, we have, as before,

$$P_n = n(n-1)(n-2)\ldots 3.2.1,$$
$$= \lfloor n .$$

Permutations where some letters are the same.

239. If certain letters, p in number, become the same, there can no longer be any distinction of order amongst them, and the $\lfloor p$ permutations they yielded before become reduced to 1. If another group of q letters become the same, their $\lfloor q$ permutations become reduced to 1, and so on, for any number of such groups. Hence, to find the number of permutations of n things when p of them are alike and also q, we must divide P_n by $\lfloor p \times \lfloor q$.

$$\text{Required number} = \frac{P_n}{\lfloor p \lfloor q \ldots} = \frac{\lfloor n}{\lfloor p \lfloor q \ldots}. \quad \text{(v)}$$

Example.—The number of permutations which can be made out of the letters of the word pedigree $= \frac{\lfloor 8}{\lfloor 3} = 8.7.6.5.4.$

Combinations.

240. Combinations are derived from permutations of n things, taken r together, by remarking that every such group of r things yield $\lfloor r$ permutations; so if ${}^rC_n$ be the symbol for the number of combinations of n things taken r together,

$${}^rC_n = \frac{{}^rP_n}{\lfloor r} = \frac{\lfloor n}{\lfloor n-r \lfloor r}. \quad \text{(vi)}$$

This may also be written,

$${}^rC_n = \frac{n(n-1)\ldots(n-r+1)}{1.2.3\ldots r}, \quad \text{(vii)}$$

and, putting $n-r = m$,

$${}^rC_{m+r} = \frac{\lfloor m+r}{\lfloor m \lfloor r}. \quad \text{(viii)}$$

Example.—The number of combinations of 6 things, a, b, c, d, e, f, taken 4 together $= \frac{6.5.4.3}{1.2.3.4} = \frac{6.5}{1.2} = 15.$

241. Since m and r enter into formula (viii) in a precisely similar manner, it appears that we may interchange them without producing any alteration of value;

$$\therefore \quad {}^{r}C_{m+r} = {}^{m}C_{m+r},$$

$$\text{or} \qquad {}^{r}C_{n} = {}^{n-r}C_{n}.$$

These are called complementary combinations.

Cor. Since there must always be an integral number of combinations of n things taken r together, it follows that

$$\frac{n.\overline{n-1}\ldots\overline{n-r+1}}{1.2.3\ldots r} \text{ always } = \text{an integer.}$$

Thus the product of any r consecutive numbers is exactly divisible by factorial r.

Combinations formed from different Classes.

242. In the formulae to be now proved we shall make use of the following principle:—If any event can happen in m ways, and another *independent* event in n ways, the two events combined can happen in mn ways.

For any one of the ways of happening of the first event can be combined with the n ways of happening of the second, and since there are m such ways, we obtain on the whole mn ways.

If three events can happen independently in m ways, n ways, and p ways respectively, they can happen together in mnp ways. And so on. This principle is of very wide application.

243. There are m consonants, and n vowels: in how many ways can a consonant and a vowel be combined?

A consonant can be taken in m ways: a vowel in n ways: by the principle a consonant *and* a vowel in mn ways.

244. The ordinary formula of permutations may be thus proved.

Given n letters, we may take a letter for the first place in n ways: when that is done, since that letter may not be repeated, we can then take a letter in $\overline{n-1}$ ways only; after that we can take one in $\overline{n-2}$ ways, then in $\overline{n-3}$ ways, and so on.

Thus we can take them for all the places in

$$n.\overline{n-1}.\overline{n-2}\ldots 3.2.1 \text{ ways.}$$

245. If there were no restriction on the repetition of the same letter, we might take the first letter n times, and the second n times, and the third n times, and so on. And all together

$$n \times n \times n \times \ldots \times n = n^n \text{ times.}$$

246. The number of permutations, out of n things, taken r together, if the same letter might be repeated, would similarly be n^r.

247. To find the number of combinations, p together out of a class of m, and q together out of a class of n.

Out of the first class we take the combination in ${}^pC_m$ ways: and out of the second class in ${}^qC_n$ ways; we can take them together in ${}^pC_m \times {}^qC_n$ ways.

248. To find in how many ways a number $\overline{m+n}$ can be divided into two sets, one of m and the other of n things.

A set of m can be formed in ${}^mC_{m+n}$ ways; after that there remain n out of which to form n things, which can be done once only. Hence the number required is

$${}^mC_{m+n} = \frac{\lfloor m+n}{\lfloor m \, \lfloor n}.$$

249. To find in how many ways a number $(m+n+p)$ can be divided into three sets, of m, n, and p things respectively.

The m things can be formed in ${}^mC_{m+n+p}$ ways: then the n things in ${}^nC_{n+p}$ ways: then the p things in ${}^pC_p$ ways, or 1 way only.

The number required is

$${}^mC_{m+n+p} \times {}^nC_{n+p} = \frac{\lfloor m+n+p}{\lfloor m \, \lfloor n+p} \times \frac{\lfloor n+p}{\lfloor n \, \lfloor p} = \frac{\lfloor m+n+p}{\lfloor m \, \lfloor n \, \lfloor p},$$

and a similar formula for more sets.

Examples.

(1) In how many ways can an election of three members for a county turn out, the number of candidates being five?

The answer is the number of combinations of five things taken three together $= \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} = 10$.

(2) If the order of the successful candidates be taken into account, the answer would be, the number of permutations of five things taken three together $= 5.4.3 = 60$.

(3) Required the number of ways of placing on a shelf three works, of two, three, and four volumes, respectively, so that the volumes belonging to the same work may be found together, but irrespective of order.

In the first place the order of the works themselves may be permuted in $3 \times 2 \times 1 = 6$ ways.

Then the volumes of the first may be permuted in 2 ways, those of the second in $3 \times 2 \times 1 = 6$ ways, and those of the third in $4 \times 3 \times 2 \times 1 = 24$ ways.

The required answer will be the product of these numbers $= 6 \times 2 \times 6 \times 24 = 1728$.

(4) In how many ways can 7 pennies be arranged on a table, as to showing heads or tails.

The first can be arranged in 2 ways: so can the second: so can all the seven. By principle in Art. 242 the answer is $2^7 = 128$.

(5) In how many ways can we make a word of three letters out of an alphabet of 26 letters, (i) if the same letters are permitted to recur, (ii) if they must be all different.

(i) There are 26 ways of taking a letter for the first place, and 26 for the second, and 26 for the third: therefore there are $26 \times 26 \times 26 = 17,576$ ways.

(ii) In this case there are 26 ways of taking a letter for the first place: but for the second we have only 25 ways, as one letter must be excluded, and for the third there are only 24 ways: hence the number of ways $= 26 \times 25 \times 24 = 15600$.

(6) At a railway station there are 10 persons waiting to take tickets. Tickets are issued for 20 different stations and of three classes: in how many ways can they take tickets?

The 10 persons, 20 stations, and 3 classes may all be regarded as independent events, hence the answer is $3 \times 10 \times 20 = 600$.

Exercises. No. 64.

(1) Find the number of permutations which can be made with 6 things.

(2) Find the number of permutations of the letters *abcd*, and write down those which have *d* standing first.

(3) Find the number of permutations of 10 things.

(4) How many changes can be rung with 5 bells?

(5) In how many ways can the letters *abcdet* be arranged, taken all together, so that the word *cat* shall appear as part of the arrangement?

(6) Find the number of permutations which can be made with the letters of the word 'possession.'

(7) Find the number of permutations which can be made with the letters of the word 'constitution.'

(8) How many words consisting of two consonants and one vowel (in the middle) can be made with the letters of the word 'importance'?

(9) How many numbers of 5 digits each can be made with 5 counters, marked with 5 different digits?

(10) How many numbers of one, two, or more digits can be made with the above counters?

(11) How many of the last set of numbers will begin with the same figure?

(12) In how many ways can a party of 7 persons be divided into sets of 3 and 4 each?

(13) Of a party of 7 persons making an excursion, one is to ride on horseback and the rest to be divided between a chaise holding four, and a gig holding two: in how many ways can this be done?

(14) In how many ways can signals be made by showing four flags, separately or in various combinations?

(15) In how many ways can signals be made with the same flags when the order of arrangement of one flag above another is taken into account?

(16) How many words containing 4 consonants and two vowels can be formed out of 6 consonants and three vowels?

(17) How many different algebraical quantities can be formed by combining a, b, c taken all together, by addition or subtraction, using $+$ and $-$?

(18) Out of a cricket-club of 20 men eleven have to be chosen to play a match. In how many ways can this be done so as to include the five best players?

(19) Find the whole number of ways in which 10 things can be distributed amongst 5 persons.

(20) Find the whole number of ways in which m things can be distributed amongst n persons.

(21) In how many ways can 6 persons form a ring, so that 2 persons may be together?

(22) A postman hands in six letters at a house, where the household consists of five persons: in how many ways can the letters be distributed?

(23) In how many ways can three black balls and two white be drawn out of a bag containing ten of each colour?

(24) In how many ways can we divide 14 balls, all different, into three sets containing respectively 2, 3, and 9 balls.

(25) In how many ways can a product of 4 factors be decomposed into products of two factors?

(26) In how many ways can a product of mn factors be decomposed into products of m factors?

(2) Binomial Theorem.

250. The Binomial Theorem, first published by Sir Isaac Newton, gives the general law of the expansion of a Binomial raised to any given power.

If we multiply together any number of binomial factors, as $(x+a)(x+b)$ and $(x+a)(x+b)(x+c)$, and examine the process, we shall find that at any stage we have a homogeneous expression,

the degree of which is the same as the number of factors, and which is arranged by descending powers of x, the first coefficient being 1, the next coefficient the sum of the last terms of the

$$\begin{array}{l} x+a \\ x+b \\ \hline x^2+ax \\ \quad bx+ab \\ \hline x^2+(a+b)x+ab \\ x+c \quad - \\ \hline x^3+(a+b)x^2+abx \\ \quad cx^2+(a+b)cx+abc \\ \hline x^3+(a+b+c)x^2+(ab+ac+bc)x+abc \end{array}$$

binomials, the next the sum of the combinations of these quantities taken two and two together, the last term being the product of all these quantities. And it is obvious that as a, b, c enter in an exactly similar manner, wherever any one combination of them enters, taken two together, &c., all other combinations of the same kind must enter also.

At every introduction of a new factor the expression rises by one dimension and all the powers of x appear in descending order. If the dimension be n, the $\overline{r+1}|^{\text{th}}$ term will contain x^{n-r}, and its coefficient will be the sum of all the combinations of $a, b, c, \ldots$ of the r^{th} dimension.

If we now make $b, c, \ldots$ all equal to a, each of these combinations will become a^r, and their number will be ${}^rC_n$, whilst the product of the binomials becomes $(x+a)^n$;

$$\therefore (x+a)^n = x^n + {}^1C_n a x^{n-1} + {}^2C_n a^2 x^{n-2} + \ldots + {}^rC_n a^r x^{n-r} + \ldots + a^n; \quad \text{(i)}$$

$$\text{or } (x+a)^n = x^n + nax^{n-1} + \ldots + \frac{n(n-1)\ldots(n-r+1)}{1.2.3\ldots r} a^r x^{n-r} + \ldots + a^n. \quad \text{(ii)}$$

Thus the expansion of $(x+a)^n$ contains $(n+1)$ terms, the first

being x^n, the second nax^{n-1}, the indices of x descending and those of a ascending by 1 at each term, so that the sum of their indices always $= n$, and the coefficient of x^{n-r} in the $(r+1)^{\text{th}}$ term being a^r multiplied by the number of combinations of n things taken r together. This is the Binomial Theorem.

251. The above needs no further demonstration.

We may however apply the method of induction, and show that (ii) being true for any index n, it will be true also for $(n+1)$.

Let $(x+a)^n = x^n + nax^{n-1} + \ldots + {}^{r-1}C_n x^{n-r-1} + {}^rC_n a^r x^{n-r} + \ldots$ from (ii).

Multiplying by $(x+a)$, we have

$$\begin{aligned}(x+a)^{n+1} &= x^{n+1} + nax^n + \ldots + {}^{r-1}C_n x^{n-r} + {}^rC_n a^r x^{n-r+1} + \ldots, \\ &\quad + ax^n + \ldots \qquad\qquad + {}^{r-1}C_n a^r x^{n-r+1} + \ldots \\ &= x^{n+1} + (n+1)ax^n + \ldots + ({}^rC_n + {}^{r-1}C_n)a^r x^{n-r+1} + \ldots\end{aligned}$$

Now $${}^rC_n + {}^{r-1}C_n = \frac{n(n-1)\ldots(n-r+2)}{1.2.3\ldots(r-1)}\left\{\frac{n-r+1}{r} + 1\right\}$$

$$= \frac{n(n-1)\ldots(n-r+2)}{1.2.3\ldots(r-1)} \times \frac{n+1}{r} = {}^rC_{n+1};$$

so that all the terms are derived from (ii) by changing n into $n+1$. But the law holds for

$$(x+a)^2 = x^2 + 2ax + a^2,$$

and for $$(x+a)^3 = x^3 + 3ax^2 + 3a^2x + a^3.$$

Hence it holds for $n = 4, 5, \ldots$, and therefore generally for all *positive integral* values of n.

Example 1. Expand $(x+a)^4$.

First term $= x^4$.

Second term $= 4ax^3$.

Third term $= \frac{4\times 3}{1\times 2}a^2x^2 = 6a^2x^2$.

Fourth term $= \frac{4\cdot 3\cdot 2}{1\cdot 2\cdot 3}a^3x = 4a^3x$.

Fifth term $= \frac{4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4}a^4x^0 = a^4$.

$$\therefore \quad (x+a)^4 = x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4.$$

Example 2. Expand $(x+a)^5$.

First term $= x^5$.

Second term $= 5ax^4$.

Third term $= \frac{5\times 4}{1\times 2}a^2x^3 = 10a^2x^3$.

Fourth term $= \frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3}a^3x^2 = 10a^3x^2$.

Fifth term $= \frac{5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4}a^4x = 5a^4x$.

Sixth term $= \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4\cdot 5}a^5x^0 = a^5$.

$\therefore\ (x+a)^5 = x^5+5ax^4+10a^2x^3+10a^3x^2+5a^4x+a^5$.

It will be observed that in both the above examples the numerical coefficients are the same taken from either end. This follows from the property given in Art. 241, ${}^rC_n = {}^{n-r}C_n$.

This remark will, however, be inapplicable if other numerical quantities enter, as in the following example.

Example 3. Expand $(2x+3a)^5$.

We have only to change x into $2a$, a into $3a$, in the last result.

$$\begin{aligned}\therefore\ (2x+3a)^5 &= (2x)^5+5\times 3a\times(2x)^4+10(3a)^2(2x)^3\\ &\qquad +10(3a)^3(2x)^2+5(3a)^4(2x)+(3a)^5,\\ &= 2^5x^5+15\times 2^4ax^4+10\times 3^2\times 2^3a^2x^3+10\\ &\qquad \times 3^3\times 2^2a^3x^2+10\times 3^4a^4x+3^5a^5,\\ &= 32x^5+240ax^4+720a^2x^3+1080a^3x^2+810a^4x+243a^5.\end{aligned}$$

Example 4. Find the fourth term of $(x+a)^7$.

It will be found more convenient to take the $(r+1)^{\text{th}}$ term as the general term rather than the r^{th}, since the former contains a^r.

The $(r+1)^{\text{th}}$ term $\quad \frac{n(n-1)\ldots(n-r+1)}{1\cdot 2\cdot 3\ldots r}a^r x^{n-r}$.

Here $n=7$, and $r+1=4$; $\therefore r=3$.

$$\begin{aligned}\text{Required term} &= \frac{7\cdot 6\cdot 5}{1\cdot 2\cdot 3}a^3x^4,\\ &= 35a^3x^4.\end{aligned}$$

The best way, after determining r, is to write down the r factors

(in this case the three factors), in the denominator, and then commencing with n (here 7) to put in the corresponding factors in the numerator: but if r is manifestly $> \frac{n}{2}$, we may use $n-r$, instead of r, in forming the coefficient.

Example 5. Find the seventh term of $(3a+5b)^9$.

Here $n = 9$, $r+1 = 7$; $\therefore r = 6$. It will be better to use $n-r = 3$, instead of 6, in forming the coefficient. But 3 must be used for r in regard to the indices.

$$\begin{aligned}\text{Required term} &= \frac{9.8.7}{1.2.3}(5b)^6(3a)^3,\\ &= 84 \times 5^6 \times 3^3 a^3 b^6,\\ &= 35437500\, a^3 b^6.\end{aligned}$$

Example 6. Expand $(x-a)^n$.

The general term is ${}^rC_n(-a)^r x^{n-r}$, the sign of which is + when r is even, and − when r is odd. Consequently the signs are alternately + and −, the first term being +, the second −, and so on. In other respects the series is unaltered from that of $(x+a)^n$; therefore

$$(x-a)^n = x^n - nax^{n-1} + \frac{n(n-1)}{1.2}a^2x^{n-2} - \ldots + {}^rC_n(-a)^r x^{n-r} \ldots$$

Example 7. Expand $(1+x)^n$ and $(1-x)^n$, and find their sum and difference.

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \ldots$$
$$\ldots + nx^{n-1} + x^n;$$

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{1.2}x^2 - \frac{n(n-1)(n-2)}{1.2.3}x^3 + \ldots;$$

the sign of the last term being + if n is even, and − if n is odd.

$$(1+x)^n + (1-x)^n = 2\left(1 + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}x^4 + \ldots\right);$$

$$(1+x)^n - (1-x)^n = 2\left(nx + \frac{n(n-1)(n-2)}{1.2.3}x^2 + \ldots\right).$$

Example 8. Find the values of these quantities when $x = 1$.

$$2^n = 1 + n + \frac{n(n-1)}{1.2} + \frac{n(n-1)(n-2)}{1.2.3} + \ldots + n + 1;$$

$$0 = 1 - n + \frac{n(n-1)}{1.2} - \frac{n(n-1)(n-2)}{1.2.3} + \ldots;$$

$$2^{n-1} = 1 + \frac{n(n-1)}{1.2} + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4} + \ldots;$$

$$2^{n-1} = n + \frac{n(n-1)(n-2)}{1.2.3} + \ldots.$$

252. The $(r+1)^{\text{th}}$ term being

$$\frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots r} a^r x^{n-r},$$

and the r^{th} term being

$$\frac{n(n-1)\ldots(n-r+2)}{1.2.3\ldots(r-1)} a^{r-1} x^{n-r+1},$$

the $(r+1)^{\text{th}}$ term may be derived from the r^{th} by multiplying the latter by $\frac{n-r+1}{r} \frac{a}{x}$.

The new coefficient by itself may be derived from the preceding by using the multiplier $\frac{n-r+1}{r}$, which is such that the denominator is r and the sum of the numerator and denominator always $(n+1)$.

These remarks are frequently practically useful in writing down successive coefficients, and they will enable us to answer the question frequently asked,—Which is the greatest term or the greatest coefficient in a given expansion? As long as the multiplier is >1, these quantities are increasing; as soon as the multiplier becomes <1, the maximum is passed. We therefore equate the multiplier to 1, and find r from the equation so formed.

If r be integral, we get two equal terms, the r^{th} and the $(r+1)^{\text{th}}$; if fractional, the term required will be the r^{th} or the $(r+1)^{\text{th}}$, according as we take for r the next greater integer, or the next less. x is supposed to be known.

Example.—Find the greatest numerical coefficient of

$$(2x+3y)^6.$$

multiplier of the r^{th} term is

$$\frac{n-r+1}{r}\cdot\frac{3}{2}, \quad \text{or} \quad \frac{6-r+1}{r}\cdot\frac{3}{2}.$$

Putting $$\frac{3(6-r+1)}{2r} = 1,$$

$$18-3r+3 = 2r,$$

$$5r = 21,$$

$$r = 4\tfrac{1}{5};$$

$r = 4$ gives the multiplier $\frac{9}{8}$, which causes increase, whilst $r = 5$ gives $\frac{3}{4}$, which causes decrease.

The fifth term has therefore the greatest coefficient

$$= 6 \times 2 \times 3^5 = 2916.$$

Exercises. No. 65.

Expand by the Binomial Theorem—

(1) $(2-x)^3$.

(2) $(x-7)^4$.

(3) $(2a-b)^5$.

(4) $(2a-7c)^7$.

(5) $(1-x)^9$.

(6) $(2-\frac{1}{y})^4$.

(7) $(1-\frac{2}{3x})^6$.

(8) $(5a-7b)^4$.

Find the terms named in the following expansions:—

(9) The fifth term in $(3y-4z)^8$.

(10) The fourth term in $(1-\frac{3y}{2z})^6$.

(11) The eighth term in $(5-\frac{z}{5})^{10}$.

(12) The ninth term in $(\frac{2}{x}-\frac{x}{4})^{10}$.

253. The Binomial Theorem may also be used for the involution

of trinomials and multinomials, by taking two terms together, as follows:—

$$\begin{aligned}\{1+x+x^2\}^4 &= \{(1+x)^4+x^2\}^4\\ &= (1+x)^4+4x^2(1+x)^3+6x^4(1+x)^2+4x^6(1+x)+x^8\\ &= 1+4x+6x^2+\ 4x^3+\ \ x^4\\ &\qquad\quad 4x^2+12x^3+12x^4+\ 4x^5\\ &\qquad\qquad\quad 6x^4+12x^5+6x^6\\ &\qquad\qquad\qquad\quad 4x^6+4x^7\\ &\qquad\qquad\qquad\qquad\quad +x^8\\ &= 1+4x+10x^2+16x^6+19x^4+16x^5+10x^6+4x^7+x^8.\end{aligned}$$

Exercises. No. 66.

Obtain the expansions of—

(1) $(3a-5+\frac{1}{a})^2$.

(2) $(1-2m+3m^2-4m^3)^2$.

(3) $(p-3q+11r)^3$.

(4) $(1+2x-3x^2)^3$.

Find the terms containing z^3 in each of the following expansions:—

(5) $(2-3z+5z^2)^4$.

(6) $(1-2z+\frac{1}{z})^5$.

254. We have now to show that the Binomial Theorem applies to negative and fractional values of the index: that is to say, that (whatever may be the value of n)

$$(1+x)^n = 1+nx+\frac{n(n-1)}{1.2}x^2+\ldots$$
$$\ldots+\frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots r}x+\ldots,$$

the term last given being the general term or $(r+1)^{\text{th}}$. The series will be interminable, except only when n is a positive integer.

We remark, in the first place, that the interpretation of negative and fractional indices was obtained (Arts. 222–225) by putting the question,—What meaning must we give to these indices in order that they may satisfy the laws of combination which obtain between positive integral indices? Conversely, any general formula depending

on positive indices ought to hold for other values of the indices. We have remarked in Art. 62 that the general forms which result from the rules of Algebra are independent of all questions of value. The extension of the Binomial Theorem to negative and fractional values of the index will be seen to be a result of this principle, when fully understood. For a full exposition of this principle, under the name of 'The principle of the permanence of equivalent forms,' we must refer to Peacock's Algebra.

255. The reasoning which we now give is Euler's.

Let $f(m)$ be used as a short method of expressing the series

$$1 + mx + \frac{m(m-1)}{1.2}x^2 + \dots,$$

whatever may be the value of m. Then

$$f(m) = 1 + mx + \frac{m(m-1)}{1.2}x^2 + \dots,$$

$$f(n) = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \dots.$$

If we multiply these together we shall have a series of ascending powers of x, the coefficients of which will depend upon m and n, the first terms of which will be

$$f(m) \times f(n) = 1 + (m+n)x + \left\{\frac{m(m-1)}{1.2} + mn + \frac{n(n-1)}{1.2}\right\}x^2 + \dots$$

$$= 1 + (m+n)x + \frac{1}{1.2}\{m^2 + 2mn + n^2 - (m+n)\}x^2 + \dots$$

$$= 1 + (m+n)x + \frac{(m+n)(m+n-1)}{1.2}x^2 + \dots.$$

Now, since these terms can all be found as far as any desired term by the simple process of multiplication, irrespectively of the values of m and n, *their form will undergo no change when m and n are restricted to be positive integers.* Consequently, if we know the form of the general term when m and n are positive integers, we know it always: but on this last supposition

$$f(m) \times f(n) = (1+x)^m \times (1+x)^n = (1+x)^{m+n},$$

$$= 1 + (m+n)x + \frac{(m+n)(m+n-1)}{1.2}x^2 + \dots$$

and the general term, the $\overline{r+1}|^{\text{th}}$,

$$= \frac{(m+n)(m+n-1)\ldots(m+n-r+1)}{1.2.3\ldots r} x^r.$$

Consequently this is always the general term, and the series is that which we have denoted by $f(m+n)$;

$$\therefore\ f(m+n) = f(m) \times f(n).$$

Similarly, $f(m+n+p) = f(m) \times f(n+p)$,

$$= f(m) \times f(n) \times f(p),$$

as before; and to any number of terms

$$f(m+n+p+\ldots) = f(m) \times f(n) \times f(p) \times \ldots.$$

Now, let $m, n, p, \ldots$ be k in number, and all $= \frac{h}{k}$; then their sum $= h$;

$$\therefore\ f(h) = f(\tfrac{h}{k}) \times f(\tfrac{h}{k}) \times \ldots (k \text{ factors});$$

$$\therefore\ (1+x)^h = \{f(\tfrac{h}{k})\}^k,$$

whence it follows that

$$(1+x)^{\frac{h}{k}} = f(\tfrac{h}{k}),$$

$$= 1 + \frac{h}{k}x + \frac{\frac{h}{k}(\frac{h}{k}-1)}{1.2}x^2 + \ldots,$$

according to the ordinary law of the Binomial Theorem.

Again, if we multiply together

$$f(m) = 1 + mx + \frac{m(m-1)}{1.2}x^2 + \ldots;$$

$$f(-m) = 1 - mx + \frac{m(m+1)}{1.2}x^2 - \ldots;$$

$$f(m)f(-m)$$

$$= 1 + (m-m)x + \left\{\frac{m(m+1)}{1.2} + \frac{m(m-1)}{1.2} - m^2\right\}x^2 + \ldots;$$

$$= 1 + 0 \cdot x + 0 \cdot x^2 + \ldots = 1.$$

This result agrees with the formula already established for all values of m, n, viz. $f(m) \times f(n) = f(m+n)$;

$$\therefore\ f(m) \times f(-m) = f(m-m);$$

$$= f(0);$$

$$= 1.$$

Consequently, $$f(-m) = \frac{1}{f(m)},$$

or $$\frac{1}{(1+x)^m} = (1+x)^{-m} = f(-m),$$

which establishes the binomial theorem for negative values of m.

256. There is one point in this reasoning which offers some difficulty. With reference to the words in italics, it may be doubted whether a change of form has not taken place in regard to all terms beyond the $(m+n+1)^{\text{th}}$ in $f(m+n)$, where m and n are integers, since the coefficients of all such terms $= 0$. In such a case, however, the form of an expression is to be sought, not where it becomes evanescent by the vanishing of a factor, but elsewhere.

We shall further meet this difficulty, by showing in one or two particular cases (Examples 1, 2) that the general law which the above reasoning professes to establish is really observed.

Example 1.—By actual evolution

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{x^3}{16} - \frac{5x^4}{128} + \ldots.$$

But the general term $\frac{n(n-1)\ldots(n-r+1)}{1.2.3\ldots r}x^r$ gives, if $r = \frac{1}{2}$,

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{x^3}{16} - \frac{5x^4}{128} + \ldots.$$

In a similar manner we may compare $\sqrt[3]{1+x}$, obtained by actual evolution (Art. 216) with the result given by the Binomial Theorem.

Example 2. By actual division

$$(1-x)^{-1} = \frac{1}{1-x} = 1 + x + x^2 + \ldots + x^r + \ldots.$$

Dividing again by $1-x$, by Horner's single process, employing x as operator,

$$(1-x)^{-2} = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots + (r+1)x^r + \ldots,$$

$$(1-x)^{-3} = \frac{1}{(1-x)^3} = 1 + 3x + 6x^2 + 10x^3 + \ldots,$$

$$(1-x)^{-4} = \frac{1}{(1-x)^4} = 1 + 4x + 10x^2 + 20x^3 + \ldots.$$

all which are in agreement with the result of the general term $\frac{n(n+1)\dots(n+r-1)}{1.2.3\dots r}x^r$, derived from writing $-n$ for n and $-x$ for x in the ordinary formula.

Example 3. Expand $(1+\frac{2x}{3})^{\frac{1}{3}}$ to 4 terms.

$$(1+\tfrac{2x}{3})^{\frac{1}{3}}$$
$$= 1+\tfrac{1}{3}\cdot\tfrac{2x}{3}+\frac{\frac{1}{3}(\frac{1}{3}-1)}{1.2}(\tfrac{2x}{3})^2+\frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}{1.2.3}(\tfrac{2x}{3})^3+\dots$$
$$= 1+\frac{2x}{9}-\frac{4x^2}{81}+\frac{40}{2187}x^3-\dots.$$

Example 4. Expand $(x-\frac{3y}{2})^{\frac{3}{2}}$ to 4 terms.

$$(x-\tfrac{3y}{2})^{\frac{3}{2}} = x^{\frac{3}{2}}-\tfrac{3}{2}x^{\frac{1}{2}}(\tfrac{3y}{2})$$
$$+\frac{\frac{3}{2}(\frac{3}{2}-1)}{1.2}\cdot x^{-\frac{1}{2}}(\tfrac{3y}{2})^2-\frac{\frac{3}{2}(\frac{3}{2}-1)(\frac{3}{2}-2)}{1.2.3}x^{-\frac{3}{2}}(\tfrac{3y}{2})^3+\dots$$
$$= x^{\frac{3}{2}}-\frac{9}{4}x^{\frac{1}{2}}y+\frac{27}{32}\frac{y^2}{x^{\frac{1}{2}}}+\frac{27}{128}\frac{y^3}{x^{\frac{3}{2}}}-\dots.$$
$$= x^{\frac{3}{2}}\left\{1-\frac{9}{4}\frac{y}{x}+\frac{27}{32}\frac{y^2}{x^2}+\frac{27}{128}\frac{y^3}{x^3}+\dots\right\}.$$

This will be found to be in general the most convenient arrangement of such expansions, as we have thus only one fractional index to write down, and the indices of y and x are the same.

Example 5. Expand $(1-x+x^2)^{\frac{1}{2}}$ to 4 terms.

$$(1-x+x^2)^{\frac{1}{2}} = 1-\tfrac{1}{2}(x-x^2)$$
$$+\frac{\frac{1}{2}(\frac{1}{2}-1)}{1.2}(x-x^2)^2-\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{1.2.3}(x-x^2)^3+\dots$$
$$= 1-\tfrac{1}{2}x+\tfrac{1}{2}x^2$$
$$-\tfrac{1}{8}x^2+\tfrac{1}{4}x^3-\dots \qquad -\tfrac{1}{16}x^3+\dots$$
$$= 1-\tfrac{1}{2}x+\tfrac{3}{8}x^2+\tfrac{3}{16}x^3+\dots.$$

Example 6. Expand $(1-3x+2x^2)^{\frac{2}{3}}$ as far as x^3.

$$(1-3x+2x^2)^{\frac{2}{3}} = 1-\frac{2}{3}(3x-2x^2)$$
$$+\frac{\frac{2}{3}(\frac{2}{3}-1)}{1.2}(3x-2x^2)^2-\frac{\frac{2}{3}(\frac{2}{3}-1)(\frac{2}{3}-2)}{1.2.3}(3x-2x^2)^3+\dots$$
$$= 1-2x+\frac{4}{3}x^2$$
$$-x^2+\frac{4}{3}x^3-\dots$$
$$-\frac{4}{3}x^3+\dots$$
$$= 1-2x+\frac{1}{3}x^2+\dots.$$

Example 7. To employ the Binomial Theorem to find the value of $\sqrt{2}$ and $\sqrt[3]{2}$, &c.

This will be done by representing 2 or other number under consideration in a form which will connect it with the highest square, cube, &c. of 100 or 1000, as the case may be.

Dividing 100 by 2, we have the quotient 50, the nearest square to which is 49;

$$\sqrt{2} = \sqrt{\frac{98}{49}} = \sqrt{\frac{100}{49}\times\frac{98}{100}},$$
$$= \frac{10}{7}\sqrt{1-\frac{2}{100}};$$
$$\therefore\ \sqrt{2} = \frac{10}{7}\left\{1-\frac{1}{2}\cdot\frac{1}{50}-\frac{1}{8}\cdot\frac{1}{(50)^2}+\frac{1}{16}\frac{1}{(50)^3}-\dots\right\},$$
$$= \frac{10}{7}\{1-\cdot01-\cdot00005+\cdot0000005-\dots\},$$
$$= \frac{1}{7}\{10-\cdot1-\cdot0005+\cdot000005-\dots\},$$
$$= \frac{1}{7}\{10\cdot000005-1\cdot0005\},$$
$$= \frac{1}{7}\times 9\cdot899505, \qquad = 1\cdot414215\dots.$$

This method may be carried very much further with great

facility, and offers the easiest way of carrying the result to a large number of decimal places.

Similarly, dividing 1000 by 2, we have the quotient 500, the nearest cube to which is 512.

$$\sqrt[3]{2} = \sqrt[3]{\frac{1024}{512}} = \sqrt[3]{\frac{1000}{512} \times \frac{1024}{1000}},$$

$$= \frac{10}{8}\sqrt[3]{1 + \frac{24}{1000}} \qquad = \frac{5}{4}\left\{1 + \frac{24}{1000}\right\}^{\frac{1}{3}},$$

$$= \frac{5}{4}\left\{1 + \frac{1}{3} \times \cdot 024 + \frac{\frac{1}{3}(\frac{1}{3} - 1)}{1.2} \times \cdot 000576 + \frac{\frac{1}{3}(\frac{1}{3} - 1)(\frac{1}{3} - 2)}{1.2.3} \times \cdot 00001382 4\right\},$$

$$= \frac{5}{4}\left\{1 + \cdot 008 - \cdot 000064 + \cdot 0000008 53\right\},$$

$$= 1 \cdot 25992106.$$

Exercises. No. 67.

Find four terms of the expansions of—

(1) $(a-x)^{\frac{1}{4}}$. (4) $(1-2x+3x^2)^{\frac{1}{2}}$.

(2) $(1-2x)^{\frac{3}{7}}$. (5) $(1-3z-z^2)^{-2}$.

(3) $(8-3y)^{\frac{2}{3}}$. (6) $(1-7z)^{-6}$.

Find the terms named in the following expansion :—

(7) The 4^{th} in $(2-5c)^{\frac{1}{2}}$. (9) The 7^{th} in $(1-8x)^{-3}$.

(8) The 9^{th} in $(2b-3d)^{-\frac{1}{2}}$. (10) The 10^{th} in $(\frac{1}{2}+\frac{x}{3})^{-6}$.

Apply the Binomial Theorem to find—

(11) The square root of 11 to 7 decimal places.

(12) The cube root of 15 to 7 decimal places.

(3) Systems of Numeration.

257. The general formula for a number of n digits in the ordinary numeration system is

$$N = a.10^{n-1} + b.10^{n-2} + \ldots + p.10 + q,$$

the highest value being $10^n - 1$ when the digits $a, b, \ldots$ all $= 9$, and the lowest 10^{n-1} when $a = 1$ and the others $= 0$. We may extend the formula to m decimal places as follows:—

$$N = a.10^{n-1} + \ldots + p.10 + 9 + \frac{r}{10} + \frac{s}{10^2} + \ldots + \frac{l}{10^m};$$

and this may be written

$$N = \frac{M}{10^m}, \text{ where } M = a.10^{m+n-1} + \ldots + l, \text{ an integer.}$$

If we have another number of m' decimal places, such as

$$N' = \frac{M'}{10^{m'}},$$

$$NN' = \frac{MM'}{10^{m+m'}},$$

and this is a decimal of $(m+m')$ places; whence we derive the common rule for the multiplication of decimals:—*Multiply together the numbers obtained by omitting the decimal points, and mark off from the right-hand as many decimal places as there are in the two decimals taken together.*

This supposes that any ciphers which may arise on the right-hand of the product MM' shall be carefully written down.

Again, if N', instead of being a fraction, be an integer closed by m' ciphers, or be of the form $N' = M'.10^{m'}$, then

$$NN' = \frac{MM'}{10^{m-m'}}, \quad \text{or} \quad MM'.10^{m'-m},$$

according as m or m' be the greater.

Again,

$$\frac{N}{N'} = \frac{M}{10^m} \times \frac{10^{m'}}{M'} = \frac{M}{M'} \cdot 10^{m'-m}, \text{ if } m' > m;$$

$$= \frac{M}{M'} \qquad \text{if } m' = m;$$

$$= \frac{M}{M'} \cdot \frac{1}{10^{m-m'}} \text{ if } m' < m:$$

whence the rule for the division of decimals:—*Remove the decimal*

points, and find the quotient of the one number so obtained divided by the other. If the number of decimal places in the divisor and dividend be equal, this is the quotient required; but if there be more decimal places in the dividend than in the divisor, place the decimal point so as to diminish the number of digits in the quotient by the excess, whilst if, on the other hand, there be more decimal places in the divisor than in the dividend, the number of figures in the quotient must be increased by the difference, supplying ciphers if necessary to that extent.

Thus, $\frac{210{\cdot}15}{70{\cdot}05} = \frac{21015}{7005} = 3$, $\frac{21{\cdot}015}{70{\cdot}05} = \frac{3}{10} = {\cdot}3$,

and $\frac{2101{\cdot}5}{70{\cdot}05} = 3 \times 10 = 30$.

These are the ordinary rules for the pointing of decimals in Multiplication and Division of Decimals.

258. If we call the index of the highest power of n in the formula for n the characteristic (or grade) of the number, we see that this will be $n-1$ for an integer of n figures, 0 for a single digit, -1 for a decimal having a significant figure in the first decimal place, and $-m$ for a decimal having its first significant figure in the m^{th} decimal place.

This method of classifying numbers by their characteristics is of great use in Logarithms. Chap. XIV.

It also furnishes a rule which is often very convenient for pointing in multiplication and division of decimals (see Grade Rule, pp. 38, 39 in *Scholar's Arithmetic*), especially for pointing from the left-hand when the decimals are interminable.

For Multiplication.—*To find the grade of any product, add the grades of the two factors, and* 1 *extra when the leading digits of the product have fallen in value.*

For Division.—*To find the grade of the quotient, subtract the grade of the divisor from that of the dividend, and* 1 *extra when the leading digits of the divisor exceed those of the dividend.*

259. Questions are sometimes asked respecting numbers, on

the supposition that some other number were used instead of 10 as the base of the numeration system. Such systems are commonly called 'scales of notation.'

Let r be such a number, called the base or radix: then the formula for a number is

$$N = ar^{n-1} + br^{n-2} + \ldots + pr + q, \quad \text{(i)}$$

or, if extended fractionally,

$$N = ar^{n-1} + \ldots + pr + q + \frac{s}{r} + \frac{t}{r^2} + \ldots. \quad \text{(ii)}$$

Since $q, p, \ldots$ are the successive overs or remainders, when N is divided repeatedly by r in (i); any number may be expressed in the new system by dividing it in the common numeration system by the new base.

Example.—Express 3150 in a system the base of which is 7:

7	3150
7	450 – 0
7	64 – 2
7	9 – 1
	1 – 2

Answer 12120.

On the other hand, to return to the common numeration system we have only to find the value of the formula (i) when $r = 7$. This is best done by Horner's single process, Art. 214, as follows, using the operator 7:

1	2	1	2	0
	9	64	450	3150 = Answer.

Example.—To express the common decimal ·5164 in a system of numeration fractions, whose base is 7. See Ex. 1 below.

It is clear that $s, t, \ldots$ are the integral values which will arise successively in (ii) if the fractional part, taken alone, be repeatedly multiplied by r (in this case 7), the integral parts as they arise being withdrawn, as in the common method of reducing any decimal of £1 to £ *s. d.* See Ex. 1.

Example 1.

	·5164
	7
3	·6148
	7
4	·3036
	7
2	·1252
	7
0	·8764
	7
6	·1348

Answer ·34206...

Example 2.

3	4	2	0	6	
	25	177	1239	8679	
		7	8679		
		7	1239·85714..		
		7	177·12244.		
		7	25·30320.		
		7	3·61474.		
			·51639..		

To return, note that

$$\frac{s}{r}+\frac{t}{r^2}+\frac{u}{r^3}=\frac{sr^2+tr+u}{r^3}.$$

Hence the rule,—*Find the value of the numerator, regarded as an integer in the same system, and divide by r as many times as there are figures in the first fraction or the numerator.* The *quotient* is the answer. See Ex. **2.**

We find the value of ·34206 (base 7) to be 8679 (base 10), and this we divide 5 times by 7, since there are 5 figures used (3, 4, 2, 0, 6) in the original fraction.

Exercises. No. 68.

(1) Express 7315 in the nonary system of numeration (base 9).

(2) Express 3277 in the octenary system (base 8).

(3) If 23543 be a number expressed in the system of numeration, whose base is 6, find its value in the common system.

(4) Find the value of the common decimal ·613 in the septenary system of numeration fractions.

(5) If ·2123 be expressed in the numeration system whose base is 5, find its value in the common system.

260. Let m be the sum of the digits of a given number.

Then $N = a.10^{n-1} + b.10^{n-2} + \ldots + p.10 + q,$

$m = a \quad + b \quad + \ldots + p \quad + q,$

$N - m = a(10^{n-1} - 1) + b(10^{n-2} - 1) + \ldots + 9p.$

Now any quantity $10^r - 1$ is divisible by $10 - 1$, or 9, whatever may be the value of r.

$\therefore$ $N - m$ is divisible by 9; and we may write

$N = 9s + m$, and any other number

$N' = 9s' + m'$,

$\therefore$ $NN' = 9^2ss' + 9ms' + 9m's + mm'$.

Consequently NN' when divided by 9 leaves the same remainder as mm' does: which leads to the common test called *Casting out the nines.*

(4) Harmonic Progression.

261. A series of terms $a, b, c, d, \ldots$ are said to be in Harmonic Progression (H. P.) if $a : c :: a - b : b - c$, the same relation holding amongst every set of three consecutive quantities.

Harmonic properties, or those which are connected with this progression, are of great importance in Geometry; but of no great interest in relation to Algebra.

The only algebraical property of importance is the following:— *The reciprocals of quantities in Harmonic Progression are in Arithmetic Progression.*

For, since $a : c :: a - b : b - c,$

$\therefore$ $ab - ac = ac - bc;$

and dividing by abc,

$$\frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a};$$

that is, $\frac{1}{c}, \frac{1}{b}, \frac{1}{a}$ have a common difference.

Cor. $$\frac{1}{a} + \frac{1}{c} = \frac{2}{b}.$$

Cor. $b = \frac{2ac}{a+c}$, which is therefore the value of a harmonic mean between a and c.

Cor. Since the n^{th} term of an A. P. $= a+(n-1)d$, the n^{th} term of a H. P. $(a, b, c, \ldots) = \dfrac{1}{\frac{1}{a}+(n-1)(\frac{1}{b}-\frac{d}{a})}$;

$$= \frac{ab}{(n-1)a-(n-2)b}.$$

There is no general expression for the sum of n terms.

To find the present value of an annuity of £a for n years, at simple interest, at the rate of £r for £1 for 1 year.

The present value will be the sum of the present values of each payment: if this be P, we have, by Art. 144,

$$P = \frac{a}{1+r}+\frac{a}{1+2r}+\frac{a}{1+3r}+\ldots\frac{a}{1+nr};$$

and this is a series in H. P. and cannot be summed by a general formula.

If a particular case be required to be summed, it is most easily done by a table of reciprocals, such as is given in Barlow's Tables.

Exercises. No. 69.

(1) Continue for 4 terms H. P. 6, 3, 2.

(2) Insert a harmonic mean between $\frac{1}{8}$ and $\frac{1}{9}$.

(3) Insert two harmonic means between $\frac{1}{3}$ and $\frac{1}{9}$.

(4) Prove that the arithmetic, geometric, and harmonic means between two quantities, a and b, form a G. P., whose common ratio is $\dfrac{2\sqrt{ab}}{a+b}$.

(5) Hence prove that the arithmetic, geometric, and harmonic means are in descending order of magnitude.

(5) Series.

262. Besides the three progressions already considered, there are many series of interest which occur in Algebra and its applications. A few of the simpler series only will now be noticed.

263. To find the sum of a series of n integral terms.

Let the general term, the n^{th}, expressed in terms of n, contain n^p as the highest power of n, and let s_n denote the sum of n terms.

We may assume

$$s_n = An + Bn^2 + \ldots + Pn^p + Qn^{p+1},$$

$A, B, \ldots P, Q$ being unknown, or, as they are called in this case, Indeterminate Coefficients, and we shall be able to obtain sufficient conditions to determine them.

The method will be best understood by an example.

We will take the ordinary A. P., $1 + 2 + \ldots + n$.

Here the highest power of n being n^1, we need only introduce n^2 into the expression for s_n.

Let $s_n = An + Bn^2$.

Then $s_{n-1} = A(n-1) + B(n-1)^2$, by writing $n-1$ for n.

$$\therefore \quad s_n - s_{n-1} = A + 2Bn - B.$$

But $s_n - s_{n-1}$ = the n^{th} term $= n$;

$$\therefore \quad A - B + 2Bn = n;$$

and this is to be made identically true. Consequently,

$$A - B = 0, \qquad 2B = 1;$$

$$\therefore \quad B = \tfrac{1}{2};$$

$$\therefore \quad A = \tfrac{1}{2};$$

$$\therefore \quad s_n = \tfrac{1}{2}(n + n^2) = \tfrac{1}{2}n(n+1).$$

It is clear that if $n = 0$ or the number of terms $= 0$, $s_n = 0$; which is the reason that it is unnecessary to place a constant term or term independent of n at the beginning of the expression for s_n.

Again, if we were to add a term containing Cn^3 or other higher term in the expression assumed for s_n, we should have on the left-hand side of the equation a term containing C as a multiplier of n^2 and no corresponding term on the other side, which implies $C = 0$; and this is the reason that it is unnecessary to introduce any higher terms than one which contains an index next above the highest index in the general term.

264. To find the sum of $1^2 + 2^2 + \ldots + n^2$.

Let $s_n \equiv An+Bn^2+Cn^3$.

Then $s_{n-1} = A(n-1)+B(n-1)^2+C(n-1)^3$.

$\therefore\ s_n - s_{n-1} = A+2Bn-B+3Cn^2-3Cn+C = n^2$.

$$\therefore\ A-B+C = 0,$$
$$2B-3C = 0,$$
$$3C = 1,$$
$$\therefore\ C = \tfrac{1}{3},$$
$$B = \tfrac{1}{2},$$
$$A = \tfrac{1}{2}-\tfrac{1}{3} = \tfrac{1}{6};$$
$$\therefore\ s_n = \tfrac{1}{6}n+\tfrac{1}{2}n^2+\tfrac{1}{3}n^3,$$
$$= \frac{n}{6}(1+3n+2n^2),$$
$$= \frac{n(1+n)(1+2n)}{1.2.3}.$$

Example. Let $n = 1$, $\therefore\ s_1 = \dfrac{1.2.3}{1.2.3} = 1$;

$n = 2$, $\therefore\ s_2 = \dfrac{2.3.5}{1.2.3} = 5$;

$n = 3$, $\therefore\ s_3 = \dfrac{3.4.7}{1.2.3} = 14$.

265. To find the sum of $1^3+2^3+\ldots+n^3$.

Let $s_n = An+Bn^2+Cn^3+Dn^4$.

$\therefore\ s_{n-1} = A(n-1)+B(n-1)^2+C(n-1)^3+D(n-1)^4$,

$$s_n - s_{n-1} = A+2Bn-B+3Cn^2-3Cn+C$$
$$+4Dn^3-6Dn^2+4Dn-D = n^3.$$
$$\therefore\ A-B+C-D = 0,$$
$$2B-3C+4D = 0,$$
$$3C-6D = 0,$$
$$4D = 1;$$
$$\therefore\ D = \tfrac{1}{4},\quad C = \tfrac{1}{2};$$
$$B = \tfrac{3}{2}C-2D = \tfrac{3}{4}-\tfrac{1}{2} = \tfrac{1}{4},$$
$$A = \tfrac{1}{4}-\tfrac{1}{2}+\tfrac{1}{4} = 0;$$
$$s_n = \tfrac{1}{4}n^2+\tfrac{1}{2}n^3+\tfrac{1}{4}n^4 = \left\{\frac{n(n+1)}{1.2}\right\}^2.$$

266. Many series which contain terms of the form

$$\frac{1}{(x+a)(x+b)}$$

may be summed by noticing that

$$\frac{1}{(x+a)(x+b)} = \frac{1}{b-a}\left\{\frac{1}{x+a} - \frac{1}{x+b}\right\}.$$

For example, to sum the series:—

$$s_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{n(n+1)},$$

$$\frac{1}{1\cdot 2} = \frac{1}{1} - \frac{1}{2},$$

$$\frac{1}{2\cdot 3} = \frac{1}{2} - \frac{1}{3},$$

$$\vdots$$

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Adding up the two sides, and noticing the terms which destroy each other, we have

$$s_n = 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

Cor. We may write this result, $s_n = \dfrac{1}{1+\frac{1}{n}}$.

Hence, if n be made infinite, or the series continued to infinity, the value of s_n becomes 1.

Exercises. No. 70.

Find the sum of the following series (to n terms) and test the result by putting $n = 3$:—

(1) $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \ldots$.

(2) $1\cdot 2 + 3\cdot 4 + 5\cdot 6 + \ldots$.

(3) $1\cdot 3 + 3\cdot 5 + 5\cdot 7 + \ldots$.

(4) $1^2 + 3^2 + 5^2 + \ldots$.

(5) $1^2 + 4^2 + 7^2 + \ldots$.

(6) $1^3 + 3^3 + 5^3 + \ldots$.

(7) $1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + 3\cdot 4\cdot 5 + \ldots$.

Find the sum of the following series to n terms and to infinity:—

(8) $\frac{1}{1\cdot3}+\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\ldots.$ (9) $\frac{1}{1\cdot3}+\frac{1}{2\cdot4}+\frac{1}{3\cdot5}+\ldots.$

(10) $\frac{1}{3\cdot7}+\frac{1}{7\cdot11}+\frac{1}{11\cdot15}+\ldots.$

CHAPTER XIII.

EQUATIONS.

In this chapter we propose to notice some points in regard to the discovery of unknown quantities that have not yet been discussed.

(1) Simultaneous Equations of the First Degree with more than two Unknown Quantities.

267. We have seen that to determine two unknown quantities we require two conditions, or in other words, when expressed algebraically, two equations. In the same way to determine three unknown quantities, it is necessary to have three equations, four equations to determine four unknown quantities, and, generally, n equations to determine n unknown quantities. Suppose we have three unknown quantities, x, y, z; and, to determine them, the three equations

$$\begin{cases} x+2y-z=2, & \text{(i)} \\ 2x-y+3z=9, & \text{(ii)} \\ 3x+y-2z=-1. & \text{(iii)} \end{cases}$$

By multiplying (i) by 2, and subtracting (ii), we find

$$5y-5z=-5.$$

Again, by multiplying (ii) by 3, and (iii) by 2, and subtracting the former from the latter, we obtain

$$5y-13z=-29,$$

whence, by subtraction, we have,

$$8z = 24, \quad \therefore \ z = 3,$$

whence $y = 2, \quad x = 1.$

We may always thus obtain by the elimination of the same unknown quantity between two pairs of equations, two simultaneous equations of two unknown quantities, to be solved in the ordinary way.

Exercises. No. 71.

Solve the equations:

(1) $3x-2y-2z = 2,$
$5x- y+3z = 23,$
$x-5y+6z = 1.$

(2) $x+y = 1,$
$x+z = 8,$
$y+z = -3.$

(3) $x-2y = 1,$
$3x+z = 14,$
$2y-3z = 7.$

(4) $3x+5y = 161,$
$7x+2z = 209,$
$2y+z = 89.$

(5) $x- y = 9,$
$y+3z = 3,$
$z-3y = 11.$

(6) $x+y+z = 28,$
$x+y-z = 2,$
$x-y+z = 14.$

(7) $x+ y+ z = 0,$
$x+2y+3z = 1,$
$x+3y+4z = 0.$

(8) $7x+ 9y+12z = 61,$
$5x+ 4y+15z = 58,$
$3x+10y+20z = 83.$

(9) Determine three numbers such that their sums, taken two and two, are 12, 13, 15.

(10) Three pipes A, B, C are employed to fill a cistern: A and B would fill it in 45 minutes, A and C in 54, B and C in 90; how long would they require separately?

(11) A sum of £13 6*s.* is paid in half-crowns, florins, and shillings: there were 10 more florins than half-crowns and shillings together, and the florins were worth 54*s.* more than the half-crowns and shillings together: how many were there of each?

(12) A's money, together with twice that of B and C, amounts to £52 10*s.*; B's, together with thrice that of A and C, to £70; and C's, together with four times that of A and B, to £82 10*s.*: required the money of each.

(2) Equations that may be solved like Quadratics.

268. Any equation which admits of being thrown into the form

$$a(\quad)^2+b(\quad)+c=0$$

may be treated as a quadratic, whatever may be the degree of the expression within the bracket. Thus one step at all events can be taken, and if the expression be of the first or second degree, the equation can be completely solved.

Example 1. To solve

$$x^2-3x+\sqrt{x^2-3x-1}=13.$$

It is obvious that by supplying -1 after the first two terms we make up a quantity corresponding to that under the radical; we have therefore, completing the square,

$$x^2-3x-1+\sqrt{x^2-3x-1}+\tfrac{1}{4}=12+\tfrac{1}{4}=\tfrac{49}{4};$$

$$\therefore\ \sqrt{x^2-3x-1}+\tfrac{1}{2}=\pm\tfrac{7}{2},$$

$$\sqrt{x^2-3x-1}=3 \text{ or } -4.$$

Squaring, $$x^2-3x-1=9 \text{ or } 16,$$

$$x^2-3x=10 \text{ or } 17,$$

whence we obtain,

$$x=5 \text{ or } -2;\quad \frac{3\pm\sqrt{77}}{2}.$$

The latter values are not applicable to the positive sign of the radical.

Example 2. $\dfrac{x^2+x+1}{x^2-x+1}+\dfrac{x^2-x+1}{x^2+x+1}=\dfrac{10}{3}.$

The second fraction is the reciprocal of the first. We obtain

$$\left(\frac{x^2+x+1}{x^2-x+1}\right)^2-\frac{10}{3}\left(\frac{x^2+x+1}{x^2-x+1}\right)+1=0,$$

$$\frac{x^2+x+1}{x^2-x+1}=3 \text{ or } \frac{1}{3}.$$

The first gives $$x^2+x+1=3(x^2-x+1),$$

$$2x^2-4x+2=0,$$

$$x=1.$$

The second gives $$3(x^2+x+1)=x^2-x+1,$$

$$2x^2+4x+2=0,$$

$$x=-1.$$

269. In a similar manner all equations may be treated which contain a number and its reciprocal. It is frequently convenient to use a symbol v to express the number or fraction, so as to replace the equation above by $v+\frac{1}{v}=\frac{10}{3}$: afterwards replacing the expression represented by v.

Exercises. No. 72.

Solve the following equations:—

(1) $x+\sqrt{x}=12.$

(2) $2\frac{x-1}{x+1}+\frac{x+1}{x-1}=\frac{17}{6}.$

(3) $x(x+1)+3\sqrt{2x^2+6x+5}=25-2x.$

(4) $x^2-7+\sqrt{3x^2-27}=110-2x^2.$

(5) $x+4+(\frac{x+4}{x-4})^{\frac{1}{2}}=\frac{12}{x-4}.$

(6) $\sqrt{\frac{x+4}{x-4}}-\sqrt{\frac{x-4}{x+4}}=1.$

270. The introduction of a symbol v is often convenient in simultaneous equations, where the unknown quantities enter symmetrically; i. e. in a precisely similar manner.

Example 1.

$$x+y=c,$$
$$y+z=a,$$
$$z+x=b.$$

Assume $x+y+z=v$:

then, adding up, $2v=a+b+c,$

$$x=v-(y+z),$$
$$=\tfrac{1}{2}(a+b+c)-a;$$
$$\therefore\ x=\tfrac{1}{2}(b+c-a).$$

Similarly,
$$y=\tfrac{1}{2}(c+a-b),$$
$$z=\tfrac{1}{2}(a+b-c).$$

Example 2.

$$x^2yz = a,$$
$$xy^2z = b,$$
$$xyz^2 = c.$$

Let $xyz = v$;

$$\therefore\ xv = a,$$
$$yv = b,$$
$$zv = c.$$

Multiplying all together,

$$xyzv^3 = abc,$$

or, $v^4 = abc$;

$$\therefore\ v = (abc)^{\frac{1}{4}},$$

$$x = \frac{a}{v} = \left(\frac{a^3}{bc}\right)^{\frac{1}{4}},\quad y = \left(\frac{b^3}{ac}\right)^{\frac{1}{4}},\quad z = \left(\frac{c^3}{ab}\right)^{\frac{1}{4}}.$$

Exercises. No. 73.

(1) $xy = 20$,
$yz = 30$,
$zx = 24$.

(2) $x(y+z) = 3$,
$y(x+z) = 4$,
$z(x+y) = 5$.

(3) $xy = c(x+y)$,
$xz = b(x+z)$,
$yz = a(y+z)$.

(4) $x(x+y+z) = a^2$,
$y(x+y+z) = b^2$,
$z(x+y+z) = c^2$.

(5) $2(y+\frac{1}{z})(z+\frac{1}{y}) = 9$,
$3(z+\frac{1}{x})(x+\frac{1}{z}) = 16$,
$6(x+\frac{1}{y})(y+\frac{1}{x}) = 49$.

(6) $ay+bx = c$,
$cx+az = b$,
$bz+cy = a$.

(3) Equations containing Radicals.

271. To solve equations which contain radicals we must generally so arrange the equation that one of the radicals may appear by itself on one side of the equation. Then by squaring both sides this radical will disappear. If others remain repeat the same steps.

Equations when thus treated will sometimes turn out in their ultimate form simple equations.

Example. $\sqrt{x-4}+\sqrt{x-19}=\sqrt{x+5}$.

Transposing, $\sqrt{x-19}=\sqrt{x+5}-\sqrt{x-4}$.

Squaring, $x-19=x+5-2\sqrt{(x-4)(x+5)}+x-4$.

Transposing, $2\sqrt{x^2+x-20}=x+20$.

Squaring, $4(x^2+x-20)=x^2+40x+400$;

which reduces to $x^2-12x=160$,

giving $x=20$ or -8.

It is necessary, as has already appeared in Exercises No. 72, to take into account both signs of the radical, in order that all values found by the solution of the equation may be applicable. If the positive sign only be taken, we must ascertain by trial which of the values found may apply.

Exercises. No. 74.

(1) $\sqrt{x+25}-\sqrt{x}=1$.

(2) $\sqrt{x+12}=2+\sqrt{x}$.

(3) $\sqrt{x+3}+\sqrt{x+6}=3\sqrt{x}$.

(4) $2\sqrt{3x+7}=9+\sqrt{2x-3}$.

(5) $\sqrt{x+2}+\sqrt{3x+10}=\sqrt{x+7}+\sqrt{3x+3}$.

(6) $2x\sqrt{x^2+a^2}+2x\sqrt{x^2+b^2}=a^2-b^2$.

(7) $2\sqrt{x+2a}-2\sqrt{y-2a}-5\sqrt{2a}=0$,

$2\sqrt{x-2a}-2\sqrt{y+2a}+3\sqrt{2a}=0$.

(4) On Surds.

We now return to the term Surd, briefly noticed in Art. 201.

272. By a Surd in the most general sense is meant a root of any order of a numerical quantity, symbolically indicated, which does not admit of having its value expressed by a finite number of figures, either as an integer, or as a fraction. Surds are classified as quadratic surds, cubic, biquadratic, of the fifth order, &c., according as they arise from the extraction of the square root, cube root, &c. Surds of different orders may be compared as to magnitude by

the reduction of their fractional indices to a common denominator. Thus, $\sqrt{3} = 3^{\frac{1}{2}}$ and $\sqrt[3]{5} = 5^{\frac{1}{3}}$ may be compared by reducing their indices to the common denominator 6.

Then $\sqrt{3} = 3^{\frac{1}{2}} = 3^{\frac{3}{6}} = (27)^{\frac{1}{6}}$,

$\sqrt[3]{5} = 5^{\frac{1}{3}} = 5^{\frac{2}{6}} = (25)^{\frac{1}{6}}$.

Since $27 > 25$, $\therefore \sqrt{3}$ is $> \sqrt[3]{5}$.

We shall have chiefly to do with *quadratic* surds, and these we must be understood to mean by the term surd, unless the contrary is stated.

Examples of quadratic surds are $\sqrt{3}$, $\sqrt{5}$, $\sqrt{17}$. A surd has a definite value, but can only be expressed numerically as an approximate (to any required degree of accuracy) in the form of a non-terminating decimal, as $\sqrt{3} = 1{\cdot}7320508$.

273. Surds are also called Irrational Quantities, and in contradistinction all numerical quantities (and the symbols which represent them) which can have their values exactly expressed by a finite number of figures are said to be Rational.

274. The fundamental property of a surd is thus expressed in Algebra:—*A surd cannot be equal to a rational quantity.*

The radicals in the last section must not be confounded with surds. We by no means assume that $\sqrt{x-4}$ and $\sqrt{x-19}$ are incapable of being expressed as integers, &c., and in fact when we substitute for x the value $x = 20$, which we obtained by solving the equation, we find $\sqrt{x-4} = \sqrt{16} = 4$, and $\sqrt{x-19} = 1$.

It is necessary to make this remark, because surds have special properties of their own, which we do not assume such radicals to possess.

275. A number is either a prime, or of the form $N = a^n b^m c^p \ldots$, where $a, b, c, \ldots$ are its prime factors, and $n, m, p, \ldots$ their indices, positive or negative.

In taking $\sqrt{N}$, every pair of like factors of any kind will yield an integral factor, so that $\sqrt{N}$ may be reduced to the form $a\sqrt{\beta}$, where a is rational, and β is the product of single prime factors. $\sqrt{\beta}$ is called an *irreducible* surd, and surds which have the same irreducible surd are called *like* surds.

276. Any number of like surds may be added or subtracted.

Thus $$3\sqrt{3}+2\sqrt{3}=5\sqrt{3},$$

$$\sqrt{32}+\sqrt{18}=4\sqrt{2}+3\sqrt{2}=7\sqrt{2}.$$

277. *The product of two unlike irreducible surds is a surd.*

Suppose $\sqrt{a}$, $\sqrt{b}$ to be such surds, a and b being primes. And, if possible, let $\sqrt{ab}=m$, a rational quantity. Therefore $ab=m^2$, which is contrary to the above analysis of the forms of numbers.

Again, suppose $a=cd$ and $b=ce$, where c, d, e are primes, then, if possible, let $\sqrt{ab}=m$,

i. e. $$\sqrt{cd\times ce}=c\sqrt{de}=m,$$

or $$\sqrt{de}=\frac{m}{c},$$

which is the same case as before, and is impossible.

278. *A surd cannot be equal to the algebraical sum of a rational quantity and a surd.*

For, if possible, let $\sqrt{a}=m+\sqrt{b}$, where $\sqrt{a}$, $\sqrt{b}$ are surds, and m a rational quantity. Then, squaring,

$$a=m^2+2m\sqrt{b}+b;$$

$$\therefore\quad \sqrt{b}=\frac{a-m^2-b}{2m},$$

a rational quantity, which is impossible.

Cor. The sum or difference of two surds cannot be a rational quantity.

279. *The sum or difference of two surds cannot be equal to a single surd.*

For, if possible, let $\sqrt{a}+\sqrt{b}=\sqrt{c}$.

Squaring, $$a+2\sqrt{ab}+b=c;$$

$$\therefore\quad \sqrt{ab}=\tfrac{1}{2}(c-a-b),$$

which is impossible.

Thus the special characteristic of surds is that they will not coalesce with other quantities, whether rational or surd.

280. *In any equality consisting of rational quantities and surds, the rational parts on the two sides are equal, and also the irrational parts.*

For, let $a+\sqrt{b}=c+\sqrt{d}$, and, if possible, let $a=c+m$; then

$$m+\sqrt{b}=\sqrt{d},$$

which is impossible.

From this property we have the remarkable result that an equation which contains a mixture of rational and surd quantities, is equivalent to two independent equations. But this cannot be affirmed, unless we know that the quantities are *really surds*, and not merely radical *in form*.

To find the square root of a binomial surd.

281. Let $a+\sqrt{b}$ be a binomial surd, consisting of the rational part a and the irrational part b. Let the square root be $\sqrt{x}+\sqrt{y}$, where *one at least* of the quantities $\sqrt{x}$, $\sqrt{y}$ must be a surd.

Then, $$\sqrt{x}+\sqrt{y}=\sqrt{a+\sqrt{b}}.$$

Squaring, $$x+2\sqrt{xy}+y=a+\sqrt{b}.$$

Then $x+y$ is rational, and $\sqrt{xy}$ is surd: since x, y are *unlike*;

$$\therefore\ x+y=a,$$
$$2\sqrt{xy}=\sqrt{b}.$$

Squaring both, $$x^2+2xy+y^2=a^2,$$
$$4xy=b;$$
$$\therefore\ x^2-2xy+y^2=a^2-b,$$
$$x-y=\sqrt{a^2-b},$$

and, from above, $$x+y=a;$$

whence x and y are known.

Example 1. To find the square root of $5+2\sqrt{6}$.

Let $$\sqrt{x}+\sqrt{y}=\sqrt{5+2\sqrt{6}},$$
$$x+2\sqrt{xy}+y=5+2\sqrt{6},$$
$$x+y=5,$$
$$2\sqrt{xy}=2\sqrt{6},$$
$$x^2+2xy+y^2=25,$$

$$4xy = 24,$$
$$x^2 - 2xy + y^2 = 1,$$
$$x - y = 1,$$
$$x + y = 5,$$
$$x = 3, \quad y = 2, \quad \text{and} \quad \sqrt{x} + \sqrt{y} = \sqrt{2} + \sqrt{3}.$$

Example 2. To find the square root of $7 + 4\sqrt{3}$.

Let
$$\sqrt{x} + \sqrt{y} = \sqrt{7 + 4\sqrt{3}},$$
$$x + 2\sqrt{xy} + y = 7 + 4\sqrt{3},$$
$$x + y = 7,$$
$$2\sqrt{xy} = 4\sqrt{3},$$
$$x^2 + 2xy + y^2 = 49,$$
$$4xy = 48,$$
$$x^2 - 2xy + 5 = 1,$$
$$x - y = 1,$$
$$x + y = 7,$$
$$x = 4, \quad y = 13,$$
$$\sqrt{x} + \sqrt{y} = 2 + \sqrt{3}.$$

282. The restriction furnished by the nature of the case that one (at least) of the unknown quantities is to be a surd, may be considered as furnishing a condition, in addition to that expressed by the equation, so preserving the validity of the rule that two conditions are required for the determination of two unknown quantities. We shall see how in the case of indeterminate equations, a limitation upon the roots, that they are to be integers, enables us to find solutions of two unknown quantities by means of one equation only.

283. From the above investigation and the two equations

$$x + y = a, \quad \sqrt{xy} = \sqrt{b},$$

we may form the following rule for finding the square root of a binomial surd. Look out amongst the factors of b for two factors whose sum is a. If two such factors can be found, they are x and y. This method, of inspection, is far the easiest method of discovering the required square root. It is necessary, before using this method, to arrange the binomial surd in the required form, viz. $a \pm 2\sqrt{b}$, the coefficient of $\sqrt{b}$ being 2.

Some simple examples of the method of dealing with surds follow.

Example 1.—To reduce $\sqrt{18}$, $\sqrt{50}$, $\sqrt{98}$ to like surds.

$$\sqrt{18} = \sqrt{9\times 2} = 3\sqrt{2}, \quad \sqrt{50} = \sqrt{25\times 2} = 5\sqrt{2},$$
$$\sqrt{98} = \sqrt{49\times 2} = 7\sqrt{2}.$$

Example 2. Arrange $5\sqrt{3}$, $17\sqrt{5}$ in the form of complete surds.

$$5\sqrt{3} = \sqrt{25\times 3} = \sqrt{75}, \quad 17\sqrt{5} = \sqrt{289\times 5} = \sqrt{1445}.$$

Example 3. Add together $\sqrt{27}+\sqrt{243}-\sqrt{48}$.

$$\sqrt{27}+\sqrt{243}-\sqrt{48} = 3\sqrt{3}+9\sqrt{3}-4\sqrt{3}.$$
$$= 8\sqrt{3}.$$

If $a+\sqrt{b}$ be a binomial surd, it can always be rationalized by multiplying it by $a-\sqrt{b}$; for $(a+\sqrt{b})(a-\sqrt{b}) = a^2-b$.

We shall call $a+\sqrt{b}$ and $a-\sqrt{b}$ Complementary Surds.

Example 4. Simplify $\dfrac{2+\sqrt{3}}{2-\sqrt{3}} - \dfrac{2-\sqrt{3}}{2+\sqrt{3}}$.

A fraction of the form $\dfrac{2+\sqrt{3}}{2-\sqrt{3}}$ may always be simplified by multiplying numerator and denominator by the complementary surd of the denominator.

$$\therefore \quad \frac{2+\sqrt{3}}{2-\sqrt{3}} - \frac{2-\sqrt{3}}{2+\sqrt{3}} = \frac{(2+\sqrt{3})^2-(2-\sqrt{3})^2}{4-3},$$
$$= 4\sqrt{3}.$$

On Rationalizing Factors.

284. Since $x^m \pm y^m = (x\pm y)(x^{m-1} \mp x^{m-2}y \pm \ldots + y^{m-1})$, if we have any binomial surd, such as $a^{\frac{p}{q}} \pm b^{\frac{r}{s}}$, which we may represent by $x \pm y$, we have only to find an index m such that x^m and y^m may both be rational, and we shall obtain a factor which will make the binomial rational. It is clear that m must be the L. C. M. of q, s.

Example.—To find a factor which will rationalize $\sqrt{2}-\sqrt[3]{3}$. Here $q=2$, $s=3$; and the L. C. M. is 6: therefore $m=6$, $x=2^{\frac{1}{2}}$, $y=3^{\frac{1}{3}}$.

$$x^{m-1}+x^{m-2}y+\ldots+y^{m-1}=(2^{\frac{1}{2}})^5+(2^{\frac{1}{2}})^4 3^{\frac{1}{3}}+(2^{\frac{1}{2}})^3 3^{\frac{2}{3}}$$
$$+(2^{\frac{1}{2}})^2 3^1+(2^{\frac{1}{2}})^1 3^{\frac{4}{3}}+3^{\frac{5}{3}},$$
$$=2^{\frac{5}{2}}+4\cdot 3^{\frac{1}{3}}+2\cdot 2^{\frac{1}{2}}3^{\frac{2}{3}}+2\cdot 3+3\cdot 2^{\frac{1}{2}}3^{\frac{1}{3}}+3\cdot 3^{\frac{2}{3}}.$$

Exercises. No. 75.

Find the square roots of the following binomial surds:—

(1) $4+2\sqrt{3}$. (3) $9+\sqrt{56}$.

(2) $16-6\sqrt{7}$. (4) $3-\frac{2}{3}\sqrt{14}$.

(5) $a-2\sqrt{a-1}$.

Simplify

(6) $\dfrac{\sqrt{7}+1}{\sqrt{7}-1}$. (8) $\dfrac{\sqrt{2}+1}{\sqrt{2}-1}+\dfrac{\sqrt{2}-1}{\sqrt{2}+1}$.

(7) $\dfrac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$. (9) $\dfrac{\sqrt{12}}{(1+\sqrt{2})(\sqrt{6}-\sqrt{3})}$.

Prove that

(10) $$\frac{2+\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}}-\frac{2-\sqrt{3}}{\sqrt{2}-\sqrt{2-\sqrt{3}}}=\sqrt{\frac{2}{3}}.$$

Find a factor which will rationalize

(11) $\sqrt[3]{3}-\sqrt[3]{2}$. (12) $\sqrt{3}+\sqrt[3]{2}$.

(5) Impossible or Imaginary Quantities.

285. In the solution of the quadratic equation

$$ax^2+bx+c=0$$

we have found that

$$x=\frac{1}{2a}\left\{-b\pm\sqrt{b^2-4ac}\right\}.$$

If $b^2-4ac=0$, the roots are equal.

If $b^2-4ac=$ a complete square, the roots are rational.

If $b^2-4ac=$ a positive quantity, not a complete square, the roots are complementary surds, which always enter as a pair:

and again, if $b^2-4ac =$ a negative quantity, we have a complementary pair of roots of the form $x = a \pm \beta\sqrt{-1}$, which are said to be impossible or imaginary values. They indicate that no positive value of x exists which will satisfy the equation.

As in the case of surds, an impossible quantity cannot be equal to the sum of a real and an imaginary quantity; and, consequently, in any equality consisting of real and impossible quantities, the real quantities on the two sides are equal, and also the impossible quantities.

The method of finding the square root of a binomial surd applies to such a quantity as $a+\beta\sqrt{-1}$.

(6) Indeterminate Equations.

286. We have seen (Art. 183) that an equation containing two unknown quantities, such as $2x+3y=25$, admits of being satisfied by every possible value of x, if we take a corresponding value of y, whether positive or negative, integral or fractional, determined from the simple equation that arises on the substitution of the assumed value of x; and so also it admits of being satisfied by every value of y, if we give equal freedom for the values of x. Thus we cannot say that any particular values of x and y are determined by the equation, and such equations are said to be indeterminate. But, if we add the restriction, that the values to satisfy the equation shall be *positive integers*, we can then find in many cases several pairs of solutions that satisfy the equation and which are serviceable for special purposes. It is in this sense only that we are said to *solve* indeterminate equations.

Example 1. Taking the equation given above, let us suppose that x and y are both of them positive integers.

Then, since $2x+3y=25$,

dividing by the least coefficient, viz. 2,

$$x+y+\frac{y}{2} = 12+\frac{1}{2},$$

$$\therefore\ x+y+\frac{y-1}{2} = 12.$$

Now, since x and y are integers, it follows that $\frac{y-1}{2}$, being $= 12-x-y$, is an integer. Call it m:

$$\therefore \frac{y-1}{2} = m,$$

$$y-1 = 2m;$$

$$\therefore y = 2m+1; \quad \text{(i)}$$

$$\therefore x+2m+1+m = 12,$$

$$x = 11-3m. \quad \text{(ii)}$$

The two equations, (i) and (ii), show that if m be any integer, subject to the condition that x and y be positive, we shall obtain solutions of the equation.

Let $m = 0$, then $x = 11$, $y = 1$.
Let $m = 1$, then $x = 8$, $y = 3$.
Let $m = 2$, then $x = 5$, $y = 5$.
Let $m = 3$, then $x = 2$, $y = 7$.

A higher value of m would make x negative: consequently these are all the solutions attainable.

It will be observed that the values of x and y form two arithmetic progressions, those of x having the common difference -3, i.e. the coefficient of y with a negative sign, and those of y the common difference $+2$, the coefficient of x.

In a similar manner any other equation may be treated.

Example 2. $3x+5y = 100.$

Dividing by the least coefficient, viz. 3,

$$x+y+\frac{2y}{3} = 33+\frac{1}{3},$$

$$x+y+\frac{2y-1}{3} = 33.$$

Assume $$\frac{2y-1}{3} = m;$$

$$\therefore 2y-1 = 3m,$$

$$2y = 3m+1.$$

Dividing by the least coefficient,

$$y = m+\frac{m+1}{2}.$$

Assume $$\frac{m+1}{2} = n,$$

$$\therefore\ m = 2n-1,$$

$$y = 2n-1+n,$$

$$= 3n-1,$$

$$x = 33-(2n-1)-(3n-1),$$

$$= 35-5n,$$

Let

$n = 1,$	$\therefore\ x = 30,$	$y = 2,$
$n = 2,$	$x = 25,$	$y = 5,$
$n = 3,$	$x = 20,$	$y = 8,$
$n = 4,$	$x = 15,$	$y = 11,$
$n = 5,$	$x = 10,$	$y = 14,$
$n = 6,$	$x = 5,$	$y = 17;$

which are all the possible solutions, unless we admit

$$x = 0,\ y = 20.$$

As before, the solutions form two progressions in A. P., the common differences being the coefficients of x and y, but with opposite signs. This property, which we readily see to be necessary, makes it easy to write down all the solutions, when one pair has been found tentatively, or to express their general formulæ.

287. The same method of solution applies to such an equation as $2x-3y = 7$, where the equation contains a different sign: but in this case the number of solutions is infinite, and the two progressions in A. P. go on increasing together.

Dividing by 2, $$x-y-\frac{y}{2} = 3+\frac{1}{2},$$

$$x-y = 3+\frac{y+1}{2},$$

Let $$\frac{y+1}{2} = m;$$

$$\therefore\ y = 2m-1,$$

$$x = 3+m+2m-1;$$

$$\therefore\ x = 3m+2.$$

Thus, $$\begin{cases} x = 5, & 8, & 11, \ldots, \\ y = 1, & 3, & 5, \ldots. \end{cases}$$

O

Exercises. No. 76.

Find the integral solutions of the equations

(1) $2x+3y=25$. (5) $6x-11y=7$.

(2) $2x-5y=9$. (6) $8x+9y=211$.

(3) $3x+11y=114$. (7) $7x+5y=94$.

(4) $5x+7y=81$. (8) $13x-5y=21$.

(9) Show that the problem, 'Find the different ways in which 25 shillings can be paid by means of half-crowns and shillings,' depends upon the solution of the indeterminate equation, $5x+2y=50$. Find the answers.

(10) Find the number of ways in which an amount of 13*s.* 6*d.* can be paid with florins and sixpences.

(11) How many different ways are there of paying 17*s.* 6*d.* with half-crowns and shillings?

(12) How many different ways are there of paying 22*s.* 6*d.* with fourpenny and threepenny pieces?

(13) Find the different ways of paying a debt of £2 12*s.* 6*d.* with half-sovereigns and half-crowns.

(14) Find a number which when divided by 5 leaves a remainder 1, and when divided by 9 leaves a remainder 6.

(15) Find a number which when divided by 7 leaves a remainder 5, and when divided by 11 leaves a remainder 6.

(16) In how many ways can *A* who has nothing but fifteen shilling pieces in his pocket settle a debt of 3*s.* 6*d.* due to *B* who has nothing but four half-crowns?

(17) Show that an equation, such as $3x+51y=191$, in which the coefficients of x and y have a common factor which does not divide the absolute term, can have no integral solution.

(7) On the Roots of Equations.

288. The values which satisfy an equation are called its roots. If we reckon imaginary roots, an equation has always as many roots as its degree.

Let α, β be the roots of the quadratic

$$ax^2+bx+c=0, \quad \text{(i)}$$

or, as we may write it,

$$x^2+\frac{b}{a}x+\frac{c}{a}=0. \quad \text{(ii)}$$

Then (Art. 195),

$$\alpha=\frac{1}{2a}(-b+\sqrt{b^2-4ac}),$$

$$\beta=\frac{1}{2a}(-b-\sqrt{b^2-4ac});$$

$$\therefore \quad \alpha+\beta=-\frac{b}{a},$$

by addition, and

$$\alpha\beta=\frac{1}{4a^2}\{b^2-(b^2-4ac)\}=\frac{c}{a},$$

whence we learn that, if the expression be written in form (ii),

I. The sum of the roots is equal to the coefficient of x with its sign changed;

II. The product of the roots is equal to the last term. This term, which is independent of x, is often called the absolute term.

Again, $$(x-\alpha)(x-\beta)=x^2-(\alpha+\beta)x+\alpha\beta$$
$$=x^2+\frac{b}{a}x+\frac{c}{a},$$

and $$ax^2+bx+c=a(x-\alpha)(x-\beta).$$

289. Let us suppose a large positive value to be given to x, and in succession smaller values, passing gradually to negative values through 0. When x is $+$ and large, the expression is $+$, since $(x-\alpha)$, $(x-\beta)$ are both $+$: we shall come at some time, if α and β are real ($\beta>\alpha$), first to β, and then to α. In each case the expression vanishes; when $x>\alpha$ and $<\beta$, the expression is negative, one factor being positive and one negative; when $x<\alpha$, the expression is positive, both factors being negative.

Similarly, if we proceed upwards from a large negative quantity to positive quantities, the passage through a root is always indicated by a change of sign: and conversely, whenever there is

a change of sign, there is a root. When α and β are impossible, the expression can never change its sign.

Thus, by giving x a succession of values 0, 1, 2, &c., and -1, -2, &c., we can discover the position of the roots. The same rule applies to equations of the third and of any degree.

Suppose for instance that a is a root of the cubic

$$x^3+bx^2+cx+d=0,$$

then,
$$a^3+ba^2+ca+d=0.$$

Subtracting,

$$(x^3-a^3)+b(x^2-a^2)+c(x-a)=0.$$

Hence the equation is divisible by $x-a$, and by dividing out the factor $x-a$, we have a quadratic,

$$x^2+ax+a^2+b(x+a)+c=0,$$

which may again be resolved into the form

$$(x-\beta)(x-\gamma)=0.$$

Thus, the cubic is equivalent to

$$(x-a)(x-\beta)(x-\gamma)=0.$$

And the expression $(x-a)(x-\beta)(x-\gamma)$ changes sign whenever x passes through a root.

Cor. If the absolute term of any equation (having a positive coefficient for the highest power of x) be negative, it has a positive root. For if $x=0$, the expression is negative; and if x be large, the expression is positive.

If the equation be a quadratic, it has also a negative root, for $\alpha\beta=-$.

If the absolute term be $+$, both roots (if real) are $+$, or both $-$, and their sign is the opposite of that of the coefficient of x.

(8) Approximation to the Roots of Equations.

290. The roots of quadratics, when non-terminable decimals, may always be derived from the general formula for the roots,

$$x=\frac{1}{2a}\{-b\pm\sqrt{b^2-4ac}\},$$

by extracting the square root of b^2-4ac to the degree of accuracy required.

291. We might also use Horner's process, which is applicable to find the roots of equations of all degrees.

Let $ax^2+bx+c=0$ (i) be the given equation, m the greatest integer in the root. Write down a, b, c as the heads of columns and use m as an operator in Horner's complete process.

$$\begin{array}{lll} a & b & b \\ & ma+b & \overset{2}{m}\overset{1}{a}+mb+c \\ & 2ma+b. & \end{array}$$

The result in the last column is the value of the left-hand expression when x is put $=m$, and if we put $x=m+y$, where y is a quantity less than 1, in the equation (i), it becomes

$$a(m+y)^2+b(m+y)+c=0,$$

or, $$ay^2+(2ma+b)y+\overset{2}{m}\overset{1}{a}+mb+c=0 \quad \text{(ii)},$$

the coefficients of which are the quantities found in the several columns (as above) after Horner's complete process.

If now, since y is a fraction, we assume it equal to $\frac{z}{10}$, the equation becomes

$$\frac{az^2}{10^2}+(2ma+b)\frac{z}{10}+\overset{2}{m}\overset{1}{a}+mb+c=0,$$

or, $$az^2+10(2ma+b)z+100(ma^2+mb+c)=0,$$

and if we can find the greatest integer in z, we may repeat the process as before, and so on, finding successive decimal figures, to any required degree of approximation.

Since the methods for square root and cube root are simply methods for the solution of the equations

$$x^2-N=0, \quad x^3-N=0,$$

the method has been already taught, under those headings (Chap. X, and *Scholar's Arithmetic*, Chap. XV); and the above transformation shows the reason for the introduction of ciphers in each column after each complete process, viz. one in the first working column, two in the second, three in the third, and so on. As in

those cases, it is more convenient to write down the absolute term c, with its sign changed, as it appears when transferred to the other side of the equation, and therefore to subtract on reaching that column, instead of adding. We thus avoid the necessity of continually repeating the negative sign. The numerical process is most conveniently arranged as in the following example.

292. To approximate to the roots of $x^2-2x-1=0$.

There is a positive root, and a negative root: and since $x=-1$, $x=0$, $x=2$, $x=3$, respectively, make the expression $+$, $-$, $-$, $+$, the negative root lies between -1 and 0, and the positive root between $+2$ and $+3$.

To avoid negative quantities suppose the equation written in the form $x^2-2x=1$, which leads to our subtracting in the last column, instead of adding.

Proceed, except in the first statement of the heads of columns, exactly as in the determination of square roots by Horner's process, introducing, to avoid decimals, two ciphers in the last column and one cipher in the preceding, to close each period. We first approximate to the positive root:—

1	-2	1 (2·41421356
	0	100
	2 0	400
	2 4	11900
	2 8 0	60400
	2 8 1	3836
	2 8 2 0	1008
	2 8 2 4	160
	2 8 2 8 0	20
1	2|8|2|8|2	

The other root is immediately derived from the property I, Art. 288. If we call the root already found a, we have

$$\alpha+\beta=+2,$$
$$\beta=2-\alpha,$$
$$=-\cdot41421356.$$

293. We now give the application to a cubic, viz.

$$x^3+3x^2-4x-10=0.$$

We find, by trial, that there is a root between 1 and 2.

1	3	−4	10 (1·895106516..
	4	0	10000
	5	500	1648000
	60	1044..	92431000
	68	165200	1932625
	76	172841	118231
	840	18056300	9368
	849	18099675	297
	858	18143075	116
	8670	18|1|4|3|9|4	
	8675		
	8680		
|1	|86|85		

The method of solving a cubic equation will be found in works on the Theory of Equations, but the above approximate method is of more practical use. It should be noted that the head of the first column is not in the solution of equations necessarily 1, but may be 2, 3, or any other number which stands as the first coefficient.

Exercises. No. 77.

State what are the signs of the roots of the following quadratics, and find, by trial, the position of their roots:—

(1) $x^2+5x-5=0$. (3) $3x^2-5x-19=0$.

(2) $x^2+16x+3=0$. (4) $2x^2-21x+5=0$.

Approximate by Horner's process to a positive root of each of the following equations (to seven decimal places):—

(5) $x^2+5x-5=0$.

(6) $3x^2-5x-19=0$.

(7) $x^3-2x-5=0$.

(8) $2x^3+9x^2+8x-499=0$.

(9) $3x^3+11x^2-5x-87=0$.

(10) $x^3+2x^2+x-150=0$.

(11) $x^3+6x^2+12x-11=0$.

(12) $64x^4+32x^3+24x^2+8x-168=0$.

CHAPTER XIV.

CONTINUED FRACTIONS.

In this and the following chapter we exemplify the application of Algebra to the discovery of methods, by means of the important subjects of Continued Fractions and Logarithms. We briefly investigate some of their properties and point out their use in calculation.

294. A continued fraction is a fraction of the form

$$a+\frac{1}{b}+\frac{1}{c}.$$

Continued fractions are more conveniently written

$$a+\frac{1}{b+}\,\frac{1}{c+}\,\frac{1}{d+}\,\frac{1}{e+}\ldots.$$

Continued fractions were first used by Lord Brounker, and have several important properties. They will only be briefly noticed here.

295. If N, M be two numbers ($N>M$) and we perform upon them the operation for the discovery of their greatest common measure, $y_1, y_2, y_3 \ldots y_n$ being the successive quotients,

$$\frac{N}{M}=y_1+\frac{1^{st}\text{ remainder}}{M}=y_1+\cfrac{1}{\cfrac{M}{1^{st}\text{ remainder}}},$$

$$\frac{M}{1^{st}\text{ remainder}}=y_2+\frac{2^{nd}\text{ remainder}}{1^{st}\text{ remainder}}=y_2+\cfrac{1}{\cfrac{1^{st}\text{ remainder}}{2^{nd}\text{ remainder}}};$$

$$\therefore\ \frac{N}{M}=y_1+\frac{1}{y_2+}\,\frac{1}{y_3+}\ldots \quad\text{and}\quad \frac{M}{N}=\frac{1}{y_1+}\,\frac{1}{y_2+}\,\frac{1}{y_3+}\ldots.$$

Example 1. To put $\frac{17}{15}$ and $\frac{15}{17}$ in the forms of continued fractions.

The successive quotients are 1, 7, 2;

$$\therefore \frac{17}{15} = 1 + \frac{1}{7+\frac{1}{2}} \text{ and } \frac{15}{17} = \frac{1}{1+\frac{1}{7+\frac{1}{2}}}.$$

Example 2. To put $\frac{119}{73}$ in the form of a continued fraction.

The successive quotients are 1, 1, 1, 1, 2, 2, 1, 2;

$$\therefore \frac{119}{73} = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{2+\frac{1}{2+\frac{1}{1+\frac{1}{2}}}}}}}.$$

296. The fractions which are obtained by stopping short at the 1st, 2nd, 3rd, ... quotients and reducing are called *converging fractions*.

They are alternately less and greater than the true value of the fraction they represent.

For y_1 is part of the quotient and not the whole. Again, to stop at y_2 makes the denominator too small, and consequently the fraction $\frac{1}{y_2}$ too great, and $\frac{N}{M}$ too great. Again, y_3 makes the 2nd denominator too small, and consequently the first too great and the whole result too small, and so on alternately.

297. We now show how to form these fractions.

Let $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots \frac{p_n}{q_n}$ represent the successive converging fractions.

$$p_1 = y_1,$$
$$q_1 = 1;$$
$$\frac{p_2}{q_2} = y_2 + \frac{1}{y_2} = \frac{y_1 y_2 + 1}{y_2} = \frac{p_1 y_2 + 1}{y_2}.$$

$\frac{p_3}{q_3}$ is formed from $\frac{p_2}{q_2}$ by writing $y_2 + \frac{1}{y_3}$ for y_2;

$$\therefore \frac{p_3}{q_3} = \frac{p_1(y_2+\frac{1}{y_3})+1}{y_2+\frac{1}{y_3}}$$
$$= \frac{p_1 y_2 y_3 + p_1 + y_3}{y_2 y_3 + 1} = \frac{y_3(p_1 y_2 + 1) + p_1}{q_2 y_3 + 1},$$
$$= \frac{y_3 p_2 + p_1}{y_3 q_2 + q_1}.$$

Similarly, $$\frac{p_4}{q_4} = \frac{p_2(y_3 + \frac{1}{y_4}) + p_1}{q_2(y_3 + \frac{1}{y_4}) + q_1},$$

$$= \frac{p_2 y_3 y_4 + p_2 + p_1 y_4}{q_2 y_3 y_4 + q_2 + q_1 y_4},$$

$$= \frac{y_4(p_2 y_3 + p_1) + p_2}{y_4(q_2 y_3 + q_1) + q_2} = \frac{y_4 p_3 + p_2}{y_4 q_3 + q_2}.$$

Let us examine whether the law here suggested can be the general law, by assuming that

$$\frac{p_{n-1}}{q_{n-1}} = \frac{y_{n-1} p_{n-2} + p_{n-3}}{y_{n-1} q_{n-2} + q_{n-3}}.$$

We shall get the next convergent by changing y_{n-1} into $y_{n-1} + \frac{1}{y_n}$.

Hence, $$\frac{p_n}{q_n} = \frac{(y_{n-1} + \frac{1}{y_n}) p_{n-2} + p_{n-3}}{(y_{n-1} + \frac{1}{y_n}) q_{n-2} + q_{n-3}},$$

$$= \frac{y_{n-1} y_n p_{n-2} + p_{n-2} + y_n p_{n-3}}{y_{n-1} y_n q_{n-2} + q_{n-2} + y_n q_{n-3}},$$

$$= \frac{y_n(y_{n-1} p_{n-2} + p_{n-3}) + p_{n-2}}{y_n(y_{n-1} q_{n-2} + q_{n-3}) + q_{n-2}},$$

$$= \frac{y_n p_{n-1} + p_{n-2}}{y_n q_{n-1} + q_{n-2}},$$ which is the same law.

Hence, if the law holds for any two successive convergents, it holds for the next, and so on for the next, and so on universally. But it does hold for the 3rd and 4th; hence universally. The first step is brought under the rule by supposing an initial fraction $\frac{1}{0}$, followed by $\frac{y_1}{1}$.

We may now write $$p_n = y_n p_{n-1} + p_{n-2},$$
$$q_n = y_n q_{n-1} + q_{n-2}.$$

By means of this property we are enabled rapidly to form the converging fractions by the following simple process:—

For a new numerator multiply the last numerator by the new quotient and add the preceding numerator: and for a new denominator multiply the last denominator by the new quotient and add the preceding denominator.

298. Multiply the first of the above equations by q_{n-1}, the second by p_{n-1}, and subtract: then,

$$p_n q_{n-1} - p_{n-1} q_n = p_{n-2} q_{n-1} - p_{n-1} q_{n-2};$$
$$\therefore\ p_n q_{n-1} - p_{n-1} q_n = -(p_{n-1} q_{n-2} - p_{n-2} q_{n-1});$$
$$\therefore\ p_{n-1} q_{n-2} - p_{n-2} q_{n-1} = -(p_{n-2} q_{n-3} - p_{n-3} q_{n-2}),$$
$$\vdots$$
$$p_3 q_2 - p_2 q_3 = -(p_2 q_1 - p_1 q_2) = -1;$$
$$\therefore\ p_n q_{n-1} - p_{n-1} q_n = (-1)^n, \quad \text{by Chain Rule.}$$

Cor. 1. All converging fractions are in their lowest terms.

For, if p_n, q_n had a common factor, it must divide the left-hand member of the above equation: but this is impossible, since the right-hand member can have no such factor.

Cor. 2. In the same way we see that successive numerators are prime to each other, and likewise successive denominators.

If we divide the last result by $q_{n-1} q_n$, we have

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^n}{q_{n-1} q_n}.$$

Hence the difference of any two convergents is the reciprocal of the product of their two denominators. As these denominators continually increase, this difference continually diminishes.

299. We shall now prove that every convergent differs less than the preceding convergent from the primary fraction.

Suppose that instead of y_n, the last quotient, we write y to express the complete value of the fractional quotient needed to close the division. Then we have

$$\frac{N}{M} = \frac{y p_{n-1} + p_{n-2}}{y q_{n-1} + q_{n-2}};$$
$$\therefore\ \frac{N}{M} - \frac{p_{n-1}}{q_{n-1}} = \frac{y p_{n-1} + p_{n-2}}{y q_{n-1} + q_{n-2}} - \frac{p_{n-1}}{q_{n-1}},$$
$$= \frac{p_{n-2} q_{n-1} - p_{n-1} q_{n-2}}{q_{n-1}(y q_{n-1} + q_{n-2})} = \frac{(-1)^n}{q_{n-1}(y q_{n-1} + q_{n-2})};$$

$$\frac{N}{M} - \frac{p_{n-2}}{q_{n-2}} = \frac{yp_{n-1}+p_{n-2}}{yq_{n-1}+q_{n-2}} - \frac{p_{n-2}}{q_{n-2}},$$

$$= \frac{y(p_{n-1}q_{n-2} - p_{n-2}q_{n-1})}{q_{n-2}(yq_{n-1}+q_{n-2})} = \frac{(-1)^{n-1}y}{q_{n-2}(yq_{n-1}+q_{n-2})};$$

and this difference is $>$ the former, since y in the numerator is >1, and q_{n-2} in the denominator $< q_{n-1}$. Hence every convergent is nearer than all preceding convergents.

300. If we wrote y_n for y in the above denominators, we should diminish the denominators and increase the fraction. Hence

$$\frac{N}{M} \sim \frac{p_{n-1}}{q_{n-1}} < \frac{1}{q_{n-1}q_n}.$$

If we wrote $y_n + 1$ for y we should increase the denominators and diminish the fraction: hence

$$\frac{N}{M} \sim \frac{p_{n-1}}{q_{n-1}} > \frac{1}{q_{n-1}(q_n + q_{n-1})};$$

and these are limits of the error in the use of any convergent instead of the original fraction.

301. The convergent $\frac{p_n}{q_n}$ is nearer to $\frac{N}{M}$ than any fraction having a denominator not greater than q_n.

For the fraction $\frac{N}{M}$ is intermediate in value to any two successive fractions, one being greater than the true value and the other less.

If possible let $\frac{p}{q}$ be a fraction nearer to $\frac{N}{M}$ than one of these, and therefore lying within the interval from

$$\frac{p_{n-1}}{q_{n-1}} \text{ to } \frac{p_n}{q_n}, \ (q \text{ not } > q_n);$$

$$\therefore \ \frac{p}{q} \sim \frac{p_{n-1}}{q_{n-1}} \text{ is } < \frac{p_n}{q_n} \sim \frac{p_{n-1}}{q_{n-1}},$$

$$\therefore \ \frac{pq_{n-1} \sim p_{n-1}q}{qq_{n-1}} < \frac{1}{q_{n-1}q_n},$$

$$\therefore \ pq_{n-1} \sim p_{n-1}q \lesssim \frac{q}{q_n};$$

which is impossible, since $\frac{q}{q_n}$ = unity, or a proper fraction.

Hence $\frac{p}{q}$ is further from $\frac{N}{M}$ than the convergent nearest to it. If this be $\frac{p_n}{q_n}$, the proposition is proved; and if it be $\frac{p_{n-1}}{q_{n-1}}$, *à fortiori* $\frac{p}{q}$ is further from $\frac{N}{M}$ than $\frac{p_n}{q_n}$, which is a nearer convergent.

(1) Application to the Solution of Indeterminate Equations.

302. Let $ax+by=c$ be an indeterminate equation; find by the method of continued fractions the fractions converging to $\frac{a}{b}$, and let that which precedes $\frac{a}{b}$ be $\frac{p_{n-1}}{q_{n-1}}$: then $\frac{p_n}{q_n}=\frac{a}{b}$, and

$$p_n q_{n-1}-q_n p_{n-1}=\pm 1,$$
$$\therefore\ aq_{n-1}-bp_{n-1}=\pm 1,$$
$$acq_{n-1}-bcp_{n-1}=\pm c,$$
$$\pm acq_{n-1}\mp bcp_{n-1}=c.$$

Hence, the equation is satisfied by

$$x=\pm cq_{n-1},\quad y=\mp cp_{n-1};$$

and, since $a(x\mp cq_{n-1})+b(y\pm cp_{n-1})=0,$

$$x=\pm(cq_{n-1}-bt),\quad y=\mp(at-cp_{n-1})$$

are general formulae for the general solutions, the upper or lower sign being taken according as the value of aq_{n-1} is + or −.

Example. $11x+19y=52.$

Here the convergent preceding $\frac{11}{19}$ is $\frac{4}{7}$, and

$$7\times 11-4\times 19=1,$$
$$\therefore\ x=5\times 27-19t,$$
$$y=11t-208.$$

The formulae are not necessarily the same as those found by the method in section (6), but they lead to the same numerical results.

(2) Application to find Approximate Fractions.

303. The method of continued fractions frequently enables us to find vulgar fractions which approximately represent fractions with large denominators or non-terminating decimals, and which are more convenient to remember or to use. Such fractions are sometimes wanted, if we desire to construct a train of wheelwork, which shall represent the movements of the heavenly bodies, &c.

Example 1.—To find a fraction which shall conveniently express the ratio of the circumference of a circle to its diameter, viz. 3·14159..

Performing the process for finding the G. C. M. of 3·14159 and 100000, we shall obtain the quotients 3, 7, 15, 1, .., and the corresponding convergents are $\frac{1}{0}, \frac{3}{1}, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \ldots$

The fraction $\frac{22}{7}$ may be used for ordinary purposes: but the fraction $\frac{355}{113}$ is a very good approximation to the real value. The inverted ratio may be remembered by writing down in a row two 1's, two 3's, and two 5's, and dividing them in the *middle* by a fraction line, thus, 113/355.

Example 2.—To find a fraction which shall conveniently represent $\sqrt{2}$.

$\sqrt{2} = 1{\cdot}4142136\ldots$

Supplying the denominator, and applying the process for finding the G. C. M., we shall obtain the following quotients, 1, 2, 2, ... and 2 perpetually.

The convergents are $\frac{1}{0}, \frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \ldots$

For ordinary purposes $\frac{7}{5}$ will often be useful, but the last fraction is far closer.

304. The most rapid method however of putting the roots of an equation in the form of a continued fraction is the following. Suppose that instead of substituting for x in equation (i), Art. 291,

$x = m+y$, where $y < 1$, we substituted $x = m + \frac{1}{z}$, where $z > 1$; we should have

$$a\frac{1}{z^2} + (2ma+b)\frac{1}{z} + ma^2 + mb + c = 0,$$

or $$a + (2ma+b)z + (ma^2 + mb + c)z^2 = 0;$$

that is to say, we should have a new equation, the coefficients of which, in reversed order, would be those formed by Horner's complete process on taking for operator the greatest integer in the root. Repeating the process, and finding the greatest integer in z, and so on, we in fact form the required continued fraction. We do not in this case transpose the absolute term.

Example. $x^2 - 3 = 0$.

The quotients are the numbers which just fail to change the sign of the last term.

(i)	1	0	-3 (1
		1	-2
		2	
(ii)	2	-2	-1 (1
		0	-1
		2	
(iii)	1	-2	-2 (2
		0	-2
		2	
(iv)	2	-2	-1 (1

(iv) being the same as (ii), it is clear that we have reached a recurring period;

$$\therefore \quad x = 1 + \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\cdots,$$

$$\therefore \quad x = \frac{19}{11} \text{ or } \frac{26}{15}.$$

Example. If the length of the (tropical) year be $365{\cdot}24224^{\mathrm{d}}$, and the moon's synodic period $29{\cdot}5305887^{\mathrm{d}}$, find a convenient fraction to represent the ratio of these numbers.

The quotients are 12, 2, 1, 2, 1, 1, 17, ...

The convergents are

$$\frac{1}{0}, \frac{12}{1}, \frac{25}{2}, \frac{37}{3}, \frac{99}{8}, \frac{136}{11}, \frac{235}{19}, \frac{4131}{334}.$$

The fraction $\frac{99}{8}$ has had some celebrity under the name of the Octa-ëteris; but the fraction $\frac{235}{19}$ far greater, as the foundation of the Metonic Cycle; 235 lunations being very nearly equal to 19 years.

305. The following example will show how the method of continued fractions may be employed to obtain approximate solutions of equations of the form $a^x = b$. Such equations are called Exponential Equations.

Example. To solve $10^x = 5$.

By trial $x = 0$ makes $10^x - 5 = -4$,

$x = 1$ " $= 5$;

therefore the value of the root is between 0 and 1. Let $x = \frac{1}{m}$;

$$\therefore \quad 10^{\frac{1}{m}} = 5,$$

$$10 = 5^m.$$

$m = 1$ makes $5^m = 5$,

$= 2$ " $= 25$.

Let $m = 1 + \frac{1}{n}$;

$$\therefore \quad 5^{1+\frac{1}{n}} = 10,$$

$$5^{\frac{1}{n}} = 2,$$

$$5 = 2^n;$$

$n = 1$ makes $2^n = 2$,

$= 2$ " $= 4$,

$= 3$ " $= 8$.

Let $n = 2 + \frac{1}{p}$;

$$\therefore \quad 5 = 2^2 . 2^{\frac{1}{p}},$$

$$\frac{5}{4} = 2^{\frac{1}{p}},$$

$$\left(\frac{5}{4}\right)^p = 2.$$

It will be found that $\left(\frac{5}{4}\right)^9 < 2$, and $\left(\frac{5}{4}\right)^{10} > 2$.

Hence $x = \frac{1}{1+}\ \frac{1}{2+}\ \frac{1}{3+}\ \frac{1}{9+} \ldots .$

The convergents are $\frac{0}{1}, \frac{1}{1}, \frac{2}{3}, \frac{7}{10}, \frac{65}{93}, \ldots .$

And $\frac{65}{93} = \cdot 69892 \ldots$, which is correct to 4 places.

The value of x which satisfies the equation $10^x = 5$ is said to be the *logarithm* of 5.

Exercises. No. 78.

Express the following fractions in the form of continued fractions, and find a series of fractions converging to them :—

(1) $\frac{196}{91}$. (2) $\frac{3161}{2131}$. (3) $\frac{45311}{7129}$. (4) $\cdot 6123$.

(5) Solve $5x + 13y = 49$ in integers.

(6) Solve $8x - 3y = 1$ in integers.

(7) If 1 metre = 39·3709 inches nearly, find a series of fractions representing the ratio of a metre to a yard.

(8) Find a convenient fraction representing the ratio of a kilometre to a mile.

(9) Find a convenient fraction to represent $\sqrt{5}$.

(10) Find a convenient fraction to represent $\sqrt[3]{7}$.

CHAPTER XV.

LOGARITHMS.

306. So soon as we have given a meaning to fractional indices, we perceive not only that it is possible to give values to x which may satisfy such an equation as $a^x = N$ in particular cases, as for instance, $10^{\frac{1}{2}} = 3{\cdot}1622777$, or $10^{\frac{1}{3}} = 2{\cdot}1544347$, which values we may have obtained by extracting the square and cube roots of 10: but that fractional values may be interpolated between $\frac{1}{2}$ and $\frac{1}{3}$, &c., which will answer to values intermediate to those on the right-hand side of these equalities. We can (in fact) to the degree of accuracy to which we carry our approximations, find the value of x answering to any particular value of N in the equation $10^x = N$. We have shown in the last section how this may be done in a particular case. The value of x answering to N in the equation $a^x = N$ is said to be the *logarithm of N to base a*, and the value of x answering to N in the equation $10^x = N$ is called the *logarithm of N to base* 10; or simply, the *logarithm* of N, or log N, 10 being the base of the only system of logarithms in common use. Logarithms are always written, not as vulgar, but as decimal fractions. Consequently, we say that log $3{\cdot}1622777 = {\cdot}5$, not $\frac{1}{2}$, and that log $2{\cdot}1544347 = {\cdot}3333333$.

Lord Napier of Merchiston was the inventor of Logarithms, but the common system of Logarithms to base 10 was introduced by Briggs.

307. Tables of logarithms are of great use in calculations, owing to the following properties.

I. Addition of logarithms answers to the multiplication of numbers.

For if $10^x = m$, (i) or $x = \log m$,

and $10^y = n$, (ii) or $y = \log n$,

multiplying (i) and (ii) together,

$$10^{x+y} = mn;$$

$$\therefore \quad x+y = \log(mn),$$

or $$\log m + \log n = \log(mn).$$

II. Subtraction of logarithms answers to division of numbers.

Again, dividing (i) by (ii),

$$10^{x-y} = \frac{m}{n}; \quad \therefore \quad x-y = \log\left(\frac{m}{n}\right);$$

or $$\log m - \log n = \log\left(\frac{m}{n}\right).$$

III. Multiplication of a logarithm answers to the involution of a number, $10^x = m$.

Raising both sides to the n^{th} power,

$$10^{nx} = m^n; \quad \therefore \quad nx = \log(m^n),$$

or $$n \log m = \log(m^n).$$

IV. Division of a logarithm answers to the evolution of a number, $10^x = m$.

Taking the n^{th} root of both sides

$$10^{\frac{x}{n}} = \sqrt[n]{m};$$

$$\therefore \quad \frac{x}{n} = \log \sqrt[n]{m},$$

or $$\frac{1}{n} \log m = \log \sqrt[n]{m}.$$

From these principles we deduce the following rules.

(i) To multiply two numbers together, look out their logarithms, and having added them together, look for their sum in the table of logarithms: the number which stands opposite to it will be the product of the numbers.

(ii) To divide one number by another subtract the logarithm of the divisor from that of the dividend: look for the difference amongst the logarithms in the table; the corresponding number will be the quotient sought.

(iii) To square or cube a number, &c. multiply its logarithm by 2, 3, or appropriate index; the result will be the logarithm of the number required.

(iv) To find the square root, cube root, or other root of a number, divide its logarithm by 2, 3, or other index of the power; the result will be the logarithm of the number required.

308. Since $10^1 = 10$, 1 is the logarithm of 10,
$10^2 = 100$, 2 „ „ 100,
$10^3 = 1000$, 3 „ „ 1000,
$10^4 = 10000$, 4 „ „ 10000;

and, generally, n is the logarithm of the number denoted by 1 in the n^{th} grade-place, followed by n ciphers.

Again, since $10^{-1} = \cdot 1$, -1 is the logarithm of $\cdot 1$,
$10^{-2} = \cdot 01$, -2 „ „ $\cdot 01$,
$10^{-3} = \cdot 001$, -3 „ „ $\cdot 001$;

and, generally, $-n$ is the logarithm of a decimal, having 1 in the n^{th} decimal place, preceded by $(n-1)$ ciphers.

The numbers named above, viz. 10, 100, 1000, 10000, &c., are only the *least* numbers which have 1, 2, 3, 4, &c. as the integral parts of their logarithms: but all numbers from 10 to 99 have 1, those from 100 to 999 have 2, those from 1000 to 9999 have 3, as the integral parts of their logarithms: thus

$$\log 15 = 1\cdot 17609, \quad \log 150 = 2\cdot 17609,$$
$$\log 1500 = 3\cdot 17609, \quad \log 15000 = 4\cdot 17609;$$

and every logarithm consists of two parts, the integral part, called the characteristic (which is the number of digits, less one) and the mantissa, or fractional part, and this last is the same for all numbers having the same digits. Consequently, in a table of logarithms the mantissa only is printed, without the characteristic, and the characteristic is supplied on inspection.

We must add, with regard to negative characteristics:

$$\log 1\cdot 5 = \cdot 17609,$$
$$\log \cdot 15 = \bar{1}\cdot 17609,$$
$$\log \cdot 015 = \bar{2}\cdot 17609.$$

Observe that the mantissa is *kept positive*, even for *negative* logarithms, the negative sign being concentrated on the characteristic, and written over it to show that it is so.

309. In the small table here given of the logarithms of the nine digits *to two places;* we see that

Nos.	Logs.
1	·00
2	·30
3	·48
4	·60
5	·70
6	·78
7	·85
8	·90
9	·95

$$\log 6 = \log 2 + \log 3,$$
$$= \cdot 30 + \cdot 48 = \cdot 78;$$
$$\log 5 = \log 10 - \log 2,$$
$$= 1 - \cdot 30,$$
$$= \cdot 70;$$
$$\log 4 = 2, \log 2 = \cdot 60;$$
$$\log 2 = \frac{1}{3}, \log 8 = \cdot 30.$$

On the other hand,

$$\log 3 = \log 9 - \log 3 = \cdot 47$$

does not quite agree with the table. This arises from the fact that two-figure logarithms cannot be accurate.

310. We should have

$$\log\left(\frac{1}{2}\right) = -\log 2,$$
$$= -\cdot 30,$$
$$= \bar{1}\cdot 70.$$

A negative logarithm is converted into a positive mantissa with negative characteristic, by taking the nought-complement of the negative mantissa.

To find $\log\left(\frac{1}{2^2}\right) = \log\left(\frac{1}{4}\right)$ multiply the above logarithm by 2.

We carry thus: twice 7 is 14, carry +1; twice −1 is −2, and adding in +1 we still have −1, the minus sign to be marked above.

$$\begin{array}{r} \bar{1}\cdot 70 \\ \underline{\quad\quad 2} \\ \bar{1}\cdot 40 \end{array}$$

311. To find $\log\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}\log\left(\frac{1}{2}\right)$.

We must divide by 2. It is done as follows: Make −1 up to −2 which is divisible by 2, considering $\bar{1}\cdot 70$ as $-2 + 1\cdot 70$, and divide accordingly.

$$2\overline{)\,\bar{1}\cdot 70}\quad \bar{1}\cdot 85$$

To find $\log \frac{1}{\sqrt[3]{3}} = \frac{1}{3}\log\frac{1}{3}$.

Here we treat $\bar{1}\cdot 52$ as $-3 + 2\cdot 52$.

$$\log\left(\frac{1}{3}\right) = -\cdot 48.$$
$$= \bar{1}\cdot 52.$$
$$\frac{1}{3}\log\left(\frac{1}{3}\right) = \bar{1}\cdot 84.$$

Exercises. No. 79.

Write down the characteristics of the logarithms of—

(1) 13. (3) 3000. (5) 7·012. (7) 3135·2.
(2) 211. (4) ·15. (6) ·003. (8) ·000052.

Write down from the Table of two-figure logarithms, Art. 309, the complete logarithms (so far as the Table will give them) of—

(9) 800. (12) $\frac{1}{500}$. (15) 25.
(10) ·004. (13) ·0007. (16) 4·9.
(11) 5,000,000. (14) ·00003. (17) 2·16.

What numbers (so far as the Table will give them) correspond to the following logarithms?—

(18) 5·30. (21) 1·90. (23) $\bar{3}$·48.
(19) 2·78. (22) $\bar{1}$·95. (24) $\bar{2}$·78.
(20) 1·85.

312. We now direct attention to the accompanying Table of four-figure logarithms (see pp. 216, 217).

This table will give the logarithm of any number to four places, and inversely, the numbers to four figures that correspond to any logarithms of four figures: but the last figure in each case may possibly be wrong by a unit.

Example 1. To find from the table the logarithm of ·2137.

The characteristic will be $\bar{1}$. Next, the mantissa.

Look for **21** in the No. column: then run along the line till you come into the column headed **3**, which is the third figure: take out $\log 2130 = \bar{1}\cdot3284$

$$\underline{\quad\quad 13}$$

$$\bar{1}\cdot3297$$

The last figure is given by a proportion. Under the head of Diff. is given the difference between two successive logarithms in the table, that is, the quantity by the addition of which we pass from the logarithm of a number having three figures to another having an additional unit. For 1 in the fourth place of a number we must add the tenth part of this, viz. here, 1·9. For 2 we must add twice as much, for 3 three times as much, and so on. Thus,

for 7 we add 13·3 or 13. We should augment the final digit, if we had five or more in the next place. Hence 1·3297 is the logarithm required.

Example 2. To find the number whose logarithm is ·3142.

We find that this lies in the table between ·3139 and ·3160.

We take the one which precedes it in value, viz. ·3139, and looking to the No. column, and also to the figure at the head of the column in which ·3139 stands, we take out log 2·06 = ·3139.

By subtracting ·3139 from ·3142, we find a Diff. 3. The Diff. for 1 in the 3rd place given at the end of the line is **21**. Therefore the Diff. for 1 in the 4th place is 2·1.

Hence, log 2·061 = ·3141, log 2·062 = ·3143, which are equidistant from our value ·3142. The former seems slightly nearer. Hence we take 2·061 as the number required.

Example 3. Find the product of ·3715 × ·8913.

$$\begin{array}{llll} \log \cdot 3710 = 1\cdot 5694 & & \log \cdot 8910 = 1\cdot 9499 & \\ \underline{\quad 5 \quad} \quad \underline{\quad 6 \quad} & & \underline{\quad 3 \quad} \quad \underline{\quad 2 \quad} & \\ \log \cdot 3715 = 1\cdot 5700 & & \log \cdot 8913 \quad 1\cdot 9501 & \\ \quad\quad 1\cdot 9501 & & & \\ \hline \quad\quad 1\cdot 5201 & & & \\ \log \cdot 331 \quad = 1\cdot 5198 & & & \\ \underline{\quad 2 \quad} \quad\quad 3 & & & \\ \log \cdot 3312 & \cdot 3312 = \text{Answer.} & & \end{array}$$

Example 4. Find the product of ·8235 × ·6154 × ·3127 × (·821)².

$$\begin{array}{llll} \log \cdot 8230 = 1\cdot 9154 & \log \cdot 6150 = 1\cdot 7889 & \log \cdot 3121 = 1\cdot 4942 & \\ \underline{\quad 5 \quad} \quad \underline{\quad 3 \quad} & \underline{\quad 4 \quad} \quad \underline{\quad 3 \quad} & \underline{\quad 7 \quad} \quad \underline{\quad 10 \quad} & \\ \log \cdot 8235 = 1\cdot 9157 & \log \cdot 6154 = 1\cdot 7892 & \log \cdot 3127 = 1\cdot 4952 & \\ \quad\quad 1\cdot 7892 & & & \\ \quad\quad 1\cdot 4952 & \log \cdot 821 = 1\cdot 9143 & & \\ \quad\quad 1\cdot 9143 & & & \\ \quad\quad 1\cdot 9143 & & & \\ \hline \quad\quad 1\cdot 2287 & & & \\ \log \cdot 1690 \quad 1\cdot 2279 & & & \\ \underline{\quad 3 \quad} \quad\quad 8 & & & \\ \log \cdot 1693 & \cdot 1693 = \text{Answer.} & & \end{array}$$

A Table of Logarithms complete for four places.

No.	0	1	2	3	4	5	6	7	8	9	Diff.
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	41
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	38
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	35
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	32
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	30
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	28
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	26
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	25
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	24
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	22
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	21
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	20
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	19
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	19
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	18
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	17
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	16
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	16
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	15
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	15
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	14
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	14
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	13
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	13
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	13
35	5441	5453	5465	5478	5490	5502	5515	5527	5539	5551	12
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	12
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	12
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	11
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	11
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	11
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	11
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	10
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	10
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	10
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	10
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	9
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	9
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	9
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	9
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	8
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	8
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	8
	0	1	2	3	4	5	6	7	8	9	

A Table of Logarithms complete for four places.

No.	0	1	2	3	4	5	6	7	8	9	Diff.
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	8
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	8
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	8
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	8
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	7
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	7
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	7
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	7
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	7
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	7
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	7
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	7
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	6
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	6
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	6
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	6
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	6
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	6
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	6
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	6
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	6
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	5
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	5
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	5
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	5
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	5
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	5
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	5
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	5
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	5
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	5
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	5
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	5
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	4
	0	1	2	3	4	5	6	7	8	9	

Example 5. Find the quotient of $\cdot 7132 \div 8\cdot 162$.

$$\log \cdot 7130 = \bar{1}\cdot 8531 \qquad \log 8\cdot 160 = \cdot 9117$$

$$\log \overline{\cdot 7132}^{2} = \overline{\bar{1}\cdot 8532}^{1} \qquad \log \overline{8\cdot 162}^{2} = \overline{\cdot 9118}^{1}$$

Subtracting, $\cdot 9118$

$$\overline{\bar{2}\cdot 9414}$$

$$\log \cdot 0873 = \overline{\bar{2}\cdot 9410}^{4}$$

$$\log \overline{\cdot 08738}^{8}$$

$\cdot 08738$ = Answer.

313. A table to four places will not suffice for any important calculations: but many small calculations may be usefully made by means of such a table: it affords a useful check on calculations otherwise made.

Use of Logarithmic Tables in Calculating Compound Interest.

314. Many questions in Compound Interest may be readily solved by the use of logarithms, in the absence of a Compound Interest Table.

Example 1. Find the amount of £321 after 5 years at 3 per cent. compound interest.

From Art. 146 we have the formula $M = PR^n$;

$$\therefore \log M = \log P + n \log R,$$
$$= \log 321 + 5 \log (1\cdot 03).$$

By the four-figure Table

$$\log 1\cdot 03 = \cdot 0128$$
$$4 \times \log 1\cdot 03 = \cdot 0512$$
$$\log 321 = 2\cdot 5065$$
$$2\cdot 5705$$

By the Table $\log 372 = 2\cdot 5705$;

$$\therefore M = £372.$$

Example 2. Find the present worth of £7132 due 11 years hence, reckoning 4 % compound interest.

Here P answers to the present worth, M to the amount due:

$$\log P = \log M - n \log R,$$
$$= \log 7132 - 11 \log 1{\cdot}04,$$
$$\log 7132 = 3{\cdot}8532, \quad \log 1{\cdot}04 = {\cdot}0170,$$
$$11 \log 1{\cdot}04 = \underline{\quad {\cdot}1870}$$
$$\overline{3{\cdot}6662}$$
$$\log 4631 = 3{\cdot}6656$$
$$\overline{7} \qquad\qquad \overline{6}$$
$$£4637 = \text{Present Worth.}$$

Example 3. Find in how many years a sum of money will double itself at 5 % compound interest.

Here, $M = 2P;$

$$\therefore \quad 2P = PR^n,$$
$$2 = R^n,$$
$$n = \frac{\log 2}{\log R} = \frac{{\cdot}3010}{{\cdot}0212};$$
$$\log n = \log(\log 2) - \log(\log R),$$
$$= \log {\cdot}3010 - \log {\cdot}0212;$$
$$\log {\cdot}3010 = \ {\cdot}4786$$
$$\log {\cdot}0212 = \bar{2}{\cdot}3263$$
$$\log 14{\cdot}2 \ = 2{\cdot}1523$$

Answer $14\frac{1}{5}$ years.

Example 4. What sum will amount to £690 in 15 years at 8 % compound interest payable quarterly?

Here $M = P(1 + \frac{r}{4})^{60}$, since compound interest must be counted afresh from every quarterly period; of which there are 60 in 15 years, and we must use $\frac{r}{4}$ instead of r.

$$\therefore \quad 690 = P(1{\cdot}02)^{60},$$

$$\log P = \log 690 - 60 \log 1{\cdot}02,$$

$\log 690 = 2{\cdot}8388$ $\qquad$ $\log 1{\cdot}02 = {\cdot}0086$

$60 \log 1{\cdot}02 = {\cdot}5160$

$\log P = 2{\cdot}3228$

$\log 210 = 2{\cdot}3222$

6 Ans. £210·3 = £210 6*s*.

Example 5. To find the Present Value of an annuity of £30 to be paid for 20 years, reckoning compound interest at 4 %.

Let P be the present value, A the annual payment, n the number of years, r the rate per £1 per annum, and $1+r = R$: then the present value of the first payment $= \frac{A}{R}$,

" second " $= \frac{A}{R^2}$,

$\vdots$

" n^{th} " $= \frac{A}{R^n}$.

Total present value

$$= \frac{A}{R}\left\{1 + \frac{1}{R} + \ldots + \frac{1}{R^{n-1}}\right\},$$

$$= \frac{A}{R}\cdot\frac{1 - \frac{1}{R^n}}{1 - \frac{1}{R}} = A\,\frac{1 - R^{-n}}{R - 1}.$$

Taking the values given above, present value

$$= 30\,\frac{1 - (1{\cdot}04)^{-20}}{{\cdot}04},$$

$\log 1{\cdot}04 = {\cdot}0170$ $\qquad$ $\log 30 = 1{\cdot}4771$

$-20 \log 1{\cdot}04 = -{\cdot}3400$ $\qquad$ $\log {\cdot}543 = \overline{1}{\cdot}7348$

$= \overline{1}{\cdot}6600$ $\qquad$ $\overline{1}{\cdot}2119$

$\log {\cdot}457 = \overline{1}{\cdot}6599$ $\qquad$ $\log {\cdot}04 = \overline{2}{\cdot}6021$

$1 - {\cdot}457 = {\cdot}543$ $\qquad$ $2{\cdot}6098$

$\log {\cdot}407 = 2{\cdot}6096$

2

Answer £407·2 = £407 4*s*.

Exercises. No. 80.

Calculate the following by the four-figure table of logarithms to such degree of accuracy as the table allows:—

(1) $\cdot 3182 \times \cdot 2103$.

(2) $\cdot 3739 \div \cdot 6814$.

(3) $\cdot 0312 \times 315 \cdot 6$.

(4) $\cdot 0913 \times 32100$.

(5) $82 \div \cdot 6149$.

(6) $\cdot 7132 \div 8194$.

(7) $3 \cdot 11 \div 84 \cdot 58$.

(8) $(3 \cdot 11)^2 \times \cdot 0152$.

(9) $(5280)^2 \times (\cdot 316)^3$.

(10) $\dfrac{1276 \times (3123)^5}{(842)^6}$.

(11) $\dfrac{(\cdot 712)^3 \times (8 \cdot 31)^2 \times (\cdot 03)^4}{(\cdot 0037)^2}$.

(12) $\dfrac{(1 \cdot 732)^4 \times (\cdot 815)^6}{(\cdot 071)^3}$.

(13) $\sqrt{\cdot 8732}$.

(14) $\sqrt[3]{\cdot 3217}$.

(15) $(\cdot 3139)^{\frac{2}{3}}$.

(16) $(\cdot 005143)^{\frac{1}{10}}$.

Find the compound interest on—

(17) £181 12*s.* for 5 years at 5 per cent.

(18) £321 6*s.* for 11 years at 4 per cent.

(19) £2149 for 13 years at $3\frac{1}{2}$ per cent.

Find the amount (at compound interest) of—

(20) £6731 after 15 years at 5 per cent.

(21) £7325 after 21 years at 4 per cent.

Find the present worth (using compound interest) of—

(22) £8121 due 5 years hence, reckoning 5 %.

(23) £329 due 11 years hence, reckoning $3\frac{1}{2}$ %.

(24) Of an annuity of £25 for 31 years, reckoning 4 %.

(25) Of an annuity of £50 for 27 years, at $3\frac{1}{2}$ %.

315. We now give some account of the method of using seven-figure Tables, of which some small specimens are subjoined.

The best seven-figure tables are Hutton's, Babbage's, Schrön's. Taylor and Walton's five-figure logarithms are extremely useful for many purposes.

(No. 1.) *Specimen Tables of Logarithms to Seven Places.*

No.	0	1	2	3	4	5	6	7	8	9	Diff.
1000	0000000	0434	0869	1303	1737	2171	2605	3039	3473	3907	434
1001	4341	4775	5208	5642	6076	6510	6943	7377	7810	8244	
1002	* 8677	9111	9544	9977	0411	0844	1277	1710	2143	2576	433
1003	0013009	3442	3875	4308	4741	5174	5607	6039	6472	6905	
1004	* 7337	7770	8202	8635	9067	9499	9932	0364	0796	1228	
1005	0021661	2093	2525	2957	3389	3821	4253	4685	5116	5548	432
1006	5980	6411	6843	7275	7706	8138	8569	9001	9432	9863	
1007	0030295	0726	1157	1588	2019	2451	2882	3313	3744	4174	431
1008	4605	5036	5467	5898	6328	6759	7190	7620	8051	8481	
1009	* 8912	9342	9772	0203	0633	1063	1493	1924	2354	2784	430

(No. 2.) *Tables of Proportional Parts.*

Diff.	0	1	2	3	4	5	6	7	8	9
434		43	87	130	174	217	260	304	347	391
433		43	87	130	173	217	260	303	346	390
432		43	86	130	173	216	259	302	346	389
431		43	86	129	172	216	259	302	345	388
430		43	86	129	172	215	258	301	344	387

(No. 3.)

No.	0	1	2	3	4	5	6	7	8	9	Diff.
6420	8075350	5418	5486	5553	5621	5689	5756	5824	5891	5959	68
6421	6027	6094	6162	6230	6297	6365	6432	6500	6568	6635	P. P.
6422	6703	6771	6838	6906	6974	7041	7109	7176	7244	7312	1 | 7
6423	7379	7447	7514	7582	7650	7717	7785	7853	7920	7988	2 | 14
6424	8055	8123	8191	8258	8326	8393	8461	8529	8596	8664	3 | 20
6425	8731	8799	8867	8934	9002	9069	9137	9204	9272	9340	4 | 27
6426	* 9407	9475	9542	9610	9678	9745	9813	9880	9948	0015	5 | 34
6427	8080083	0151	0218	0286	0353	0421	0488	0556	0624	0691	6 | 41
6428	0759	0826	0894	0961	1029	1096	1164	1232	1299	1367	7 | 48
6429	1434	1502	1569	1637	1704	1772	1840	1907	1975	2042	8 | 54
											9 | 61
											67

(No. 4.)

No.	0	1	2	3	4	5	6	7	8	9
101	0043214	047512	051805	056094	060380	064660	068937	073210	077478	08174[2]
102	086002	090257	094509	098756	103000	107239	111474	115704	119931	124154
103	128372	132587	136797	141003	145205	149403	153598	157788	161974	166155
104	170333	174507	178677	182843	187005	191163	195317	199467	203613	207755
105	211893	216027	220157	224284	228406	232525	236639	240750	244857	248960
106	253059	257154	261245	265333	269416	273496	277572	281644	285715	289777
107	293838	297895	301948	305997	310043	314085	318123	322157	326188	330214
108	334238	338257	342273	346285	350293	354297	358298	362295	366289	370279
109	374265	378248	382226	386202	390173	394141	398106	402066	406023	409977

* In the lines thus marked the third figure of the logarithm changes, and this must be carefully noted.

316. In the specimens (No. 1) and (No. 3) four figures of the number are placed in the No. column, a fifth at the head of the column, so that the logarithm for 5 places is taken direct out of the tables. The difference for the other two places is obtained by a proportion from the difference given in the Diff. column, which is the difference for 1 in the fifth place.

317. At the end of specimen No. 3 in the Diff. column is a Table of Proportional Parts as usually arranged. This is formed from the difference 6·8, the difference for the 6th place, by multiplying by 1, 2, 3, ... 9 in succession, omitting the decimals and augmenting the final, when the next figure rejected is 5, or over. The same table serves for the 7th place by supposing an imaginary line to be drawn down the table of proportional parts, cutting off the last figures, but augmenting the final. No. 2 gives the tables of Proportional Parts as arranged in Chambers' Reprint of Hutton's Logarithms. This work has the fault that the less legible modern figures are used instead of those with heads and tails.

Example 1. To find the logarithm of 100·6732.

log 100·67	= 2·0029001	
3		130
2		9
log 100·6732	= 2·0029140	

Example 2. To find the number whose logarithm is ·8079231.

	·8079234	
log 6·4257	= ·8079204	
		30
4		27
4		3
log 6·425744	Answer 6·425744	

We write down the logarithm next below that given and subtract it from the latter: finding difference 30. Looking in the table which belongs to the proper difference 432, we find 27, answering to 4, as the next lower difference, and subtracting we

have 3: which again answers most nearly to 4 in the imaginary table made by cutting off the last column of figures and augmenting the final.

Exercises. No. 81.

Find from the specimen tables to 7 places the logarithms of—

(1) 10·00735. (3) 64·22783. (5) ·6425301.

(2) 1002·543. (4) 6·429371. (6) ·1008721.

Find the numbers whose logarithms are—

(7) 2·0026324. (9) ·8075432. (11) ·8079873.

(8) 1·0033872. (10) ·8075763. (12) 2·8080134.

(13) Construct the tables of proportional parts answering to the differences 436, 438.

Find from table (No. 4), using also, when necessary, the four-figure table—

(14) The amount of £21 17*s.* 6*d.* at $2\frac{1}{2}$ per cent. compound interest for 72 years.

(15) The amount of £13 3*s.* at $1\frac{3}{4}$ per cent. compound interest for 55 years.

(16) After how many years will a sum of money double itself at $2\frac{1}{4}$ per cent. compound interest?

(17) Find the present worth of an annuity of £21 for 35 years at $4\frac{1}{4}$ per cent. compound interest.

The Scholar may with advantage work out the Examples given under Chain Rule in the *Scholar's Arithmetic*, p. 259, and other like Examples by means of the four-figure Table of Logarithms.

ANSWERS TO THE EXERCISES.

ANSWERS TO THE EXERCISES.

Exercises, No. 1. Page 4.

(1) $15a$.
(2) $32b$.
(3) $27b$.
(4) $119c$.
(5) $1\frac{1}{4}d$.
(6) $\frac{8}{15}e$.
(7) $\frac{11}{21}f$.

Exercises, No. 2. Page 5.

(1) $2a+\frac{1}{2}a+\frac{2}{3}a=3\frac{1}{6}a$. (2) $3x+4x+\frac{1}{4}x=7\frac{1}{4}x$.

(3) $3a$, $5a$, $2\frac{1}{2}a$, $10\frac{1}{2}a$.

Exercises, No. 3. Page 6.

(1) $2a+3b+c$.
(2) $3a+4c+6d$.
(3) $4c+3e+2$.
(4) $5d+7$.
(5) $5a+4c+d+3$.
(6) $7a+5b+7$.
(7) $a+b$.
(8) $x+y$.
(9) $a+7$.
(10) $x+5$.
(11) $3a+15$.
(12) $2\frac{1}{3}x+10$.

(13) $25+a$, $25+\frac{3a}{2}$.

Exercises, No. 4. Page 6.

(1) 13.
(2) 13.
(3) $6+3e$.
(4) 7.
(5) 22.
(6) 38.

Exercises, No. 5. Page 7.

(1) $3a-2b$. (5) $21-2x$. (9) $4\frac{2}{3}x$.
(2) $5b-17$. (6) $3d-2c$. (10) $4x-15$.
(3) $14a-11b$. (7) $2a-3$. (11) $8x-3a$.
(4) $35-c$. (8) $3a-\frac{1}{2}b$. (12) $a-5x$.

Exercises, No. 6. Page 8.

(1) $6bc$. (2) $8064abc$. (3) $74ab$.
(4) $6ab-6$. (5) $5ax+11$.

Exercises, No. 7. Page 9.

(1) $5\frac{a}{b}$. (3) $\frac{12c}{5d}$. (5) $\frac{x}{3a}$.
(2) $\frac{ab}{2c}$. (4) $\frac{a}{6}$. (6) $\frac{ax}{y}$.

Exercises, No. 8. Page 11.

(1) a^2b. (4) a^{15}. (7) b^{18}. (10) 1.
(2) a^2b^2. (5) b^6. (8) $a^{16}b^6$. (11) a^{m+n+p}.
(3) a^3b^2c. (6) b^8c^4. (9) bc^8. (12) a^{a+b}.
(13) a^{p+q}.

Exercises, No. 9. Page 15.

(1) Seven a square minus five b square plus three c square.

(2) Three ab cube minus two abc square plus four bc cube.

(3) Seven bc to the fourth minus five c cube d square plus d to the fifth.

(4) Eight x to the fifth minus three x cube y square plus two xy to the fourth minus y to the fifth.

(5) Three ax to the fifth minus four by to the fifth plus cx square y cube.

Exercises, No. 10. Page 15.

(1) $5a+9c-3d$.
(2) $15c-17d+18e$.
(3) $81b-13e+14f-19g$.
(4) $17x+20y-13z$.
(5) $3a^2+4ab+6b^2$.
(6) $7a^2-6ab+9b^2$.
(7) $5a^2b-3ab^2+6b^3$.
(8) $5a^4+4a^3-3a^2b+7ab^3+9b^4$.

Values for No. (9):

(1) 91. (2) 392. (3) 947. (4) 235. (5) 296.

Values for No. (10):

(1) 42. (2) 99. (3) -9. (4) -89. (5) 75. (6) 63. (7) 102. (8) 771.

Exercises, No. 11. Page 17.

(1) $18a$. (2) $-17b$. (3) $8x$. (4) $5y$.
(5) $-11az$. (6) $-11a^2b$. (7) 0.

Exercises, No. 12. Page 22.

(1) $2ax-3by-5ax+by$.
(2) $6a^2x-13by^2+5abxy+9y^4+7x^3-8y^3$.
(3) $2-5+8y-13z-3-16y+15x$.
(4) $2-5+\frac{8}{3}y-7z-6-16y+3x$.
(5) $\frac{3}{2}ax-\frac{5}{2}y-2y+\frac{5x}{2a}+\frac{7z}{b}-\frac{3y}{b}$.
(6) $3a-5b-(16c-8d+3f-4h)$,
$3a-5b-16c+(8d-3f+4h)$,
$3a-5b-16c+8d-(3f-4h)$.

Exercises, No. 13. Page 23.

(1) $6bx-4ab+12b-24ab+2b^2+18bc-4bx+14ab+6b$.
(2) $-3+24+32a-20b+12c-28a+12b+8c$.
(3) $-3a-3b+6a-9b-15a-21b+48c+39a-6b+9c-45d$.
(4) $-\frac{2}{3}a+\frac{2}{3}b-3a+4b+12a-\frac{18}{7}b+\frac{10}{7}c$.

Exercises, No. 14. Page 25.

(1) $7a-7b-19c$.
(2) $6x-3y-8$.
(3) $2xy+13yz-9$.
(4) $-2+12x+4y$.
(5) ab^2+38cd.
(6) $18yz^2+42zx^2-6y^3$.
(7) $27ab+bc+38ca$.

Exercises, No. 15. Page 26.

(1) $(3a+2b+2c)x^2+(a+b-c)x+2c-3d+e$.
(2) $(2a+c+5f)x^2+(4b-2h-3)xy+(d+l+4)y^2$.
(3) $(a+c+e+2)yz+(b-d+2f+5)z^2$.

Exercises, No. 16. Page 27.

(1) $a-2b$.
(2) $a+3b-4c$.
(3) $-a-2b+3c$.
(4) $16ax+3by+3cz$.
(5) $3ab^3-2a^2b^2+2ab^3$.
(6) $-ax^3+16bx^2y-21cxy^2+25dy^3$.
(7) $-5x^3+20y^3$.
(8) $\frac{1}{6}x^3-2\frac{5}{7}x^2y+\frac{5}{9}xy^2-\frac{10}{11}y^3$.
(9) $-7x+3y-3$.

Exercises, No. 17. Page 27.

(1) $(2a-3b)x^2+(3b-5e)x+5c-6f$.
(2) $(3b-2a)x+(5a-b)y-cz-3d$.
(3) $(a-d)x^3+ex^2y-(f+g)y^3$.

Exercises, No. 18. Page 28.

(1) -6.
(2) 30.
(3) $-6cbc$.
(4) $-\frac{1}{3}x^2y$.
(5) $15a^2b^2c$.
(6) $-35a^5b^3c^2$.

Exercises, No. 19. Page 29.

(1) x^2+4x+4.
(2) x^2+6x+9.
(3) $x^2+12x+36$.
(4) $a^2+6ab+9b^2$.
(5) $b^2+6ab+9a^2$.
(6) $4a^2+20ab+25b^2$.
(7) $49a^2-42ab+9b^2$.
(8) $x^2-10xb+25b^2$.
(9) $y^2-10yz+25z^2$.
(10) $9z^2-18zm+9m^2$.

Exercises, No. 20. Page 30.

(1) $x-5$. (3) $x+11$. (5) $2x+5y$.
(2) $x-7$. (4) $y-2x$. (6) $7x-3y$.

Exercises, No. 21. Page 30.

(1) $(x+8)(x-8)$.
(2) $(2y+9)(2y-9)$.
(3) $(6+5x)(6-5x)$.
(4) $(3+3y+2x)(3-3y-2x)$.
(5) $3y(8-3y)$.
(6) $(5+2y-3z)(5-2y+3z)$.
(7) $(a-b+c)(a-b-c)$.
(8) $4y(z-x)$.

Exercises, No. 22. Page 31.

(1) x^2+5x+6.
(2) $a^2+ab+ac+bc$.
(3) $a^2+10a+21$.
(4) $x^2+14x+45$.
(5) $y^2+11y+18$.
(6) $4x^2+16x+15$.
(7) $9x^2+21xy+10y^2$.
(8) $25x^2+25xz+6z^2$.
(9) $15z^2+16xz+4x^2$.
(10) $21x^2+20xy+4y^2$.

Exercises, No. 23. Page 31.

(1) $a^2+8a+15$.
(2) x^2+3x+2.
(3) $4x^2+8x+3$.
(4) $9x^2+15xy+4y^2$.

Exercises, No. 24. Page 31.

(1) $x^2-4x-21$.
(2) $a^2-2ab-15b^2$.
(3) $x^2-8xy+15y^2$.
(4) $b^2-2bc-35c^2$.
(5) $21+4y-y^2$.
(6) $15z^2+2xz-x^2$.

Exercises, No. 25. Page 32.

(1) $(x+3)(x+5)$.
(2) $(x+3)(x+7)$.
(3) $(x+4)(x-6)$.
(4) $(x+1)(x-21)$
(5) $(x+17)(x-3)$.
(6) $(x+6)(x-7)$.
(7) $(x+y)(x-13y)$.
(8) $(x-y)(x+11y)$.

Exercises, No. 26. Pages 33 to 35.

(1) $16x^2y^2+2xy^2-38xyz-3y^2+16yz-5z^2$.

(2) $14x^5y-31x^4y^2+27x^3y^3-53x^2y^4+25xy^5-30y^6$.

(3) x^5-1.

(4) $x^{12}+x^8-x^4-1$.

(5) $3a^4bc-9a^3bc^2-2a^3b^2c+12a^3c^3+6a^2b^2c^2-8a^2bc^3$
$+4ab^4c-12b^4c^2+16b^3c^3$.

(6) $42a^5b+22a^4b^2+37a^3b^3+12a^2b^4+10ab^5-35b^6$.

(7) $56y^5-85y^4+110y^3-78y^2+58y-45$.

(8) $1+\frac{1}{8}x^2-\frac{1}{8}x^3-\frac{1}{64}x^5$.

(9) $x^6+2x^3y^3+y^6$.

(10) $a^6-2a^3b^3+b^6$.

(11) $21l^2-83lm+22m^2-27l+99m$.

(12) $3a^2-ab-10b^2+23\frac{1}{2}ac+52bc-31\frac{1}{2}c^2$.

(13) $15a^3-14a^2b+24ab^2-7b^3$.

(14) $-6a^5+a^4b+19a^3b^2-\frac{7}{2}a^2b^3-\frac{1}{2}ab^4$.

(15) $a^4+4a^3b+4a^2b^2+2b^2c^2-b^4-c^4$.

(16) a^4+4a^2+16.

(17) $a^4+4a^3b+4a^2b^2-4b^4$.

(18) $a^3+b^3+c^3-3abc$.

(19) $a^3+b^3+c^3+d^3-3abc-3bcd-3cda-3dab$.

(20) $196a^{10}c^4-36a^4b^2c^4+12a^2bc^5-c^6$.

(21) $x^3+(a+b+c)x^2+(ab+bc+ca)x+abc$.

(22) $x^3+(a-b-c)x^2-(ab-bc+ca)x+abc$.

(23) x^3+6x^2-x-30.

(24) $x^3+6x^2+12x+8$.

(25) x^3-3x^2+3x-1.

(26) $a^3+b^3+c^3+3ab^2+3a^2b+3ac^2+3a^2c+3bc^2+3b^2c+6abc$.

(27) $a^3-b^3-c^3+3ab^2+3ac^2-3a^2b-3a^2c-3bc^2-3b^2c+6abc$.

(28) $l^4-\frac{1}{l^4}$.

(29) $-a^4-b^4-c^4+2a^2b^2+2b^2c^2+2c^2a^2$.

(30) $a^3-b^3-c^3-d^3+a^2c-ac^2-ab^2+a^2b-ad^2+a^2d-3bd^2$
$3b^2d+bc^2+b^2c+c^2d+cd^2+2abc-2abd+2acd+2bcd$.

(31) $n^2p+n^2q+mnp+mnq+mpq+npq+np^2+mp^2$.

(32) m^8-n^8.

(33) $a^6+a^5+a^3+a-1$.

(34) $p^6+p^4q^2-p^2q^4-q^6+4p^4r^2+8p^2q^2r^2+4q^4r^2-16p^2r^4+16q^2r^4-64r^6$.

Exercises, No. 27. Pages 35, 36.

(1) $x^2-5xy+2y^2$.

(2) $-9a^2b+\frac{1}{3}^4ab^2-3bc^2$.

(3) $xy-20yz+4z^2$.

(4) $-30\frac{1}{8}x^3+17xz^2-13z^3$.

Exercises, No. 28. Pages 37, 38.

(1) $x^2-3xy+2y^2$.

(2) $4x^2-x+1$.

(3) $6x^3+x^2+x$.

(4) $-b+2b^2-b^3$.

(5) k^7+k^4+k.

(6) $3x^2+x-5$.

(7) x^4-4x^2+16.

(8) x^2-5.

(9) x^2-2x-7.

(10) a^6-5a-2.

(11) x^2-1.

(12) $a^3+6a^2+12a+8$.

(13) $y^2-3yz+z^2$.

(14) $x-\sqrt{x}+4$.

(15) $x^3-x\sqrt{x}+8$.

(16) $a^5-3a^3+2a^2-1$.

(17) $x^5+2x^4\sqrt{x}+2x^4-4x^3-8x^2\sqrt{x}-8x^2+16x+32\sqrt{x}+32$ with remainder -128. But read $+64$ in question, when there will be no remainder.

(19) $x^7+2x^6+4x^5+8x^4+16x^3+32x^2+64x+128$, $x^6+4x^4+16x^2+64$, and x^4+16.

(20) x^2+x-1.

(21) $2a^2-3a+4$.

(22) $3a^4-2a^3+5a^2$.

(23) $4m^2+5m-2$.

(24) $4p^2-5p+2$.

(25) $7l^3-3l^2-l$.

(26) $x^2-x\sqrt{2}+1$.

(27) $6+a$.

(28) $x^4-4x^3+10x^2-12x+9$.

(29) $a^2+\frac{2}{3}$.

(30) $6x-6$.

(31) $y^4+\frac{y^3}{3}+\frac{y^2}{9}+\frac{y}{27}+\frac{1}{81}$.

(32) $x-\frac{2}{3}$.

(33) $1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}$.

(34) $1+\frac{x}{3}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{81}+\frac{x^5}{243}$.

(35) $1-3x+3x^2-x^3$.

(36) $x^6-x^5+x^3-x+1$ and x^4-x^2+1.

(37) $x^4-2x^3+6x^2-6x+9$ and x^2+3.

Exercises, No. 29. Page 40.

(1) $x^3+ax^2+a^2x+a^3+\frac{2a^4}{x-a}$. (2) $x^3-ax^2+a^2x-a^3+\frac{2a^4}{x+a}$.

(3) $x^4+x^3y+x^2y^2+xy^3+y^4$.

(4) $x^4-x^3y+x^2y^2-xy^3+y^4-\frac{2y^5}{x+y}$.

(5) $x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5+\frac{2y^6}{x-y}$.

(6) $x^8-x^7y+x^6y^2-x^5y^3+x^4y^4-x^3y^5+x^2y^6-xy^7+y^8-\frac{2y^9}{x+y}$.

(7) x^2+2x+4. (8) x^3-2x^2+4x-8.

(9) $x^4+3x^3+9x^2+27x+81$.

(10) $27-9y+3y^2-y^3+\frac{1}{3}y^4-\frac{1}{9}y^5+\frac{1}{27}y^6-\frac{28y^7}{27(3+y)}$.

Exercises, No. 30. Page 41.

(1) a^2+ab+b^2.

(2) $a^6-a^5b+a^3b^3-ab^5+b^6$.

(3) $bx+ay$.

(4) $a-b+c-d$.

(5) $\frac{1}{a}-\frac{b}{ax}-\frac{b^2}{a^2x}-\frac{b^2}{ax^2}$.

(6) $x^4-(n-1)a^2x^2+a^4$.

(7) x^2-ax+b^2.

(8) mx^3+nx^2+p.

(9) $2c^2+3bc-b^2$.

(10) $\frac{3}{2}x^3-5x^2+\frac{x}{4}+9$.

(11) $1+an+a^2n^2$.

(12) $a^4+a^3z+az^2+z^4$.

(13) $12x^2-5xy-3y^2$.

(14) $a^2+3ab+c^2$.

(15) $16a^4-8a^3b+4a^2b^2-2ab^3+b^4$.

(16) $a^2-5ab+6b^2$.

(17) $2a+3x$.

(18) a^3+a^2-2a.

(19) $a^4-a^3-6a^2+2a+4$.

(20) cr^2+dr+c. Dividend becomes 1221; divisor 11; and quotient 111, which is verified by division.

Exercises, No. 31. Page 43.

(1) 1st quotient; $x^3+6x^2+18x+56+\frac{183}{x-3}$.

2nd quotient; $x^2+7x+25+\frac{81}{x-1}+\frac{183}{(x-1)(x-3)}$.

(2) 1st quotient;

$$3x^5+15x^4+77x^3+385x^2+1925x+9625+\frac{48118}{x-5}.$$

2nd quotient; $3x^4+21x^3+119x^2+623x+3171$

$$+\frac{15967}{x-2}+\frac{48118}{(x-2)(x-5)}.$$

3rd quotient; $3x^3+15x^2+89x+445+\frac{2281}{x+2}$

$$+\frac{15967}{(x-2)(x+2)}+\frac{48118}{(x+2)(x-2)(x-5)}.$$

(3) 1st quotient;

$$7x^7-7x^6+7x^5-7x^4+7x^3-9x^2+9x-9+\frac{14}{x+1}.$$

2nd quotient; $7x^6+7x^5+21x^4+35x^3+77x^2+145x$

$$+299+\frac{589}{x-2}+\frac{14}{(x-2)(x+1)}.$$

3rd quotient; $7x^5-7x^4+35x^3-35x^2+147x-149$

$$+\frac{597}{x+2}+\frac{589}{(x+2)(x-2)}+\frac{14}{(x+2)(x-2)(x+1)}.$$

(4) $3x^4+8x^3y+16x^2y^2+35xy^3+70y^4+\frac{146y^5}{x-2y}.$

Exercises, No. 32. Pages 45, 46.

(1) $x+\frac{a}{3}.$

(2) $x+\frac{3a}{b}.$

(3) $y+\frac{3ax-c}{2b}.$

(4) $a-\frac{b^2}{a}.$

(5) $-b+\frac{a^2}{b}.$

(6) $-3a^2+2b^2+\frac{7a^3}{b}.$

(7) $\frac{3ac+2b}{c}.$

(8) $\frac{3adx-2bdy+c}{d}.$

(9) $\frac{3axy-by^2+2}{y}.$

(10) $\frac{8bxy-7ax^2-cy}{x}.$

(11) $\frac{8b^2x-3a^2+2ab}{x}.$

(12) $\frac{6a}{b}.$

(13) $-\frac{12bc}{7}$.

(14) $\frac{2axy}{3}$.

(15) $\frac{12a^2+9ab}{2b^3}$.

(16) $\frac{b}{cd}$.

(17) $-\frac{2ad}{5c}$.

(18) $\frac{5bx}{6z}$.

(19) $\frac{3a+b}{4b^4}$.

(20) $\frac{5ac}{51d}$.

(21) $-\frac{4y}{5a}$.

(22) $\frac{3y^2z}{5cx}$.

(23) $\frac{7ax+y}{2cx}$.

(24) $\frac{3a}{4bxy^2}$.

(25) $\frac{128abx^2y^2z^2}{3c^2}$.

(26) $\frac{a^2x}{byz}$.

(27) $-\frac{bxz}{9y}$.

(28) $\frac{2a^2yz}{b^2x}$.

(29) $-\frac{3bx^2}{2ay^2z}$.

Exercises, No. 33. Page 47.

(1) $\frac{4a+3b}{12}$.

(2) $\frac{4a+3b}{ab}$.

(3) $\frac{a^2+4b^2}{2ab}$.

(4) $\frac{ad+9bc}{3bd}$.

(5) $\frac{ayz+bzx+cxy}{xyz}$.

(6) $\frac{2bcx+3cay+5abz}{abc}$.

(7) $\frac{4a-3b}{12}$.

(8) $\frac{3b-4a}{ab}$

(9) $\frac{10yz-5xz+2xy}{xyz}$.

(10) $\frac{5z-3y}{yz}$.

(11) $\frac{7b^3-15ac^2}{b^2c^2}$.

(12) $\frac{2bz+2cy-9c^2}{bc}$

(13) $\frac{90cyz-5axz+3bxy}{15xyz}$.

Exercises, No. 24. Pages 48, 49.

(1) $\frac{a^4dyz^2}{3cx}$.

(2) $\frac{ab(x^2+y^2)+(a^2+b^2)xy}{abxy}$.

(3) $\frac{x^3+ax^2-a^2x-a^3}{a^3x^3}$.

(4) $\frac{b^2-a^2-ac-bc}{ac^2-bc^2}$.

(5) $\frac{ac^2npx^2}{bmy^4}$.

(6) $\frac{a}{3}$.

(7) $x-a$.

(8) $\frac{a^4b^4}{a^4+a^2b^2+b^4}$.

Exercises, No. 35. Page 50.

(1) $\frac{b}{ab-1}$. (3) $-\frac{3}{x^2}(x+1)$. (5) x^2+ax+a^2.

(2) $\frac{bc-1}{abc-a-c}$. (4) $\frac{y+z}{x}$. (6) b.

(7) $\frac{cd-af}{ce-bf}$. (8) $\frac{(2a^2-b^2)(a-b)}{2a}$.

Exercises, No. 36. Page 52.

(1) $3ab$, $18a^2b^2$.
(2) $2xyz$, $90x^3y^3z$.
(3) $16xy^3z$, $96x^2y^4z^2$.
(4) $3axyz$, $1887a^2x^3y^3z^2$.
(5) ab, $15a^2b^3$.
(6) xy, $441a^3x^3y^3$.
(7) $3xy$, $405pqx^2y^3$.
(8) 1, $mnpxyz$.
(9) xyz, $mnpx^2y^2z^2$.
(10) ax^2yz. $87360a^4x^8y^8z^8$.

Exercises, No. 37. Page 52.

(1) $x-a$, $6(x^2-a^2)$.
(2) 2, $8(x-a)^2(x+a)$.
(3) $x+b$, $3(x+b)^2(x-b)$.
(4) $x-a$, $(x^3-a^3)(x+a)$.
(5) 1, $pqr(m^2-n^2)$.
(6) x^2+ax+a^2, $3(x^3-a^3)$.
(7) 1, $30(x^6-b^6)$.
(8) 1, $126(x^4-a^4)$.
(9) $x+2a$, $(x^2-4a^2)(x+7a)$.
(10) $x-3a$, $(x-3a)^2(x+4a)$.

Exercises, No. 38. Page 56.

	H. C. F.	L. C. M.
(1)	$x+2$;	$(x+2)(2x-5)(3x+7)$.
(2)	$3x+5$,	$(3x+5)(7x-9)(4x-5)$.
(3)	$2a-5b$,	$(2a-5b)(3a+7b)(4a+9b)$.
(4)	$2x^2-6xy-20y^2$,	$3(2x^2-6xy-20y^2)$.
(5)	$3x-5y$,	$(3x-5y)(3x+7y)(7x-2y)$.
(6)	$3x^2-5$,	$(2x-7)(3x+5)(3x^2-5)$.
(7)	x^2+2x+3,	$(2x+5)(3x+7)(x^2+2x+3)$.
(8)	$2x^2-x-5$,	$(3x+4)(x-7)(2x^2-x-5)$.
(9)	x^2+x-11,	$(x+11)(3x-1)(x^2+x-11)$.

	H. C. F.	L. C. M.
(10)	$x^2-2xy+y^2$,	$(x+5y)(x-5y)(x^2-2xy+y^2)$.
(11)	x^3+x^2+x-1,	$(x-1)(x+1)(x^3+x^2+x-1)$.
(12)	$2x^3-x+1$,	$(3x-1)(4x+1)(2x^3-x+1)$.
(13)	x^2-2a^2,	$(x^2-2a^2)(x^2+3a^2)(2x^2+5a^2)$.
(14)	x^2+3x-5,	$(2x^2-x+3)(x^2-x+1)(x^2+3x-5)$.
(15)	* $2x^2-x-1$,	$(2x^2-x-1)(2x^2+x+1)(3x^2+x-1)$.
(16)	$2x^2-x-2$,	$(2x^2-x-2)(3x^2+x+2)(4x^2+x-5)$.
(17)	$2x^2-3x+5$,	$(2x^2-3x+5)(2x^2+2x+3)(3x^3+2x-3)$.
(18)	$3x^2-x+1$,	$x(x^2+1)(x^2+x-1)(3x^2-x+1)$.
(19)	$x^4-x^3+x^2-2x+1$,	x^5-x^2-x+1.
(20)	x^3-2x^2+1,	$x^4-x^3-2x^2+x+1$.

Exercises, No. 39. Page 57.

	H. C. F.	L. C. M.
(1)	$x-1$,	$(x-1)^2(x+1)(2x+3)(3x+1)$.
(2)	$2x-3$,	$(x-1)(x+1)(2x-3)(3x-2)$.
(3)	$3x-5$,	$(3x-5)(3x+2)(x+2)(x-7)$.
(4)	$7x^2-5$,	$(3x-4)(3x+4)(7x^2-5)(2x-3)$.
(5)	$x-3$,	$(x-3)^2(x+3)^2$.
(6)	$x-5y$.	$(x+2y)(x+7y)(x-3y)(x-5y)$.

Exercises, No. 40. Page 58.

(1) $\dfrac{7a^2-3bc}{5a-a^4b^2c-4}$.

(2) $\dfrac{2a^2x^3}{3b^2}$.

(3) $\dfrac{a^2+ax+x^2}{a-x}$

(4) $\dfrac{a+y}{a^2-y}$.

(5) $\dfrac{3a+5b}{3c-\frac{2}{5}}$.

(6) $\dfrac{x+3}{x+6}$.

(7) $\dfrac{9x^2-x-3}{x+5}$.

(8) $\dfrac{2x^2+3x-5}{x(x+2)}$.

(9) $\dfrac{a+b+c}{a-b-c}$.

(10) $\dfrac{a-2b}{a+b-c}$.

(11) $\dfrac{x-7}{x+7}$.

(12) $\dfrac{4x+3}{3x^2-5x+7}$.

* Dele $-4x$ in the second expression.

Exercises, No. 41. Page 59.

(1) $\frac{4xy}{x^2-y^2}$.

(2) $\frac{2x-3}{(x-1)^2(x-2)}$.

(3) $\frac{2-x}{x^3-1}$.

(4) $\frac{-6x}{x^4+x^2+1}$.

(5) $2\frac{x^2-x+1}{1-x^3}$.

(6) $\frac{2a+3}{x^2-a^2}$.

(7) $\frac{2ab+2bc-2ac+a^2-3b^2+c^2}{(a-b+c)(a+b-c)(a-b-c)}$.

(8) $\frac{2x(16-3x)}{(x-4)(x-6)(x-8)}$.

(9) $\frac{(x-1)(x-5)}{(x+1)(x-2)(x-3)}$.

(10) $\frac{a^3-4a^2x-11ax^2-2x^3}{2x(a^2-x^2)}$.

(11) $\frac{2(5x-1)}{1-4x^2}$.

(12) $\frac{a^3+ab^2+b^3}{(a+b)^3}$.

(13) $\frac{1}{x^2-1}$.

(14) $\frac{1}{(x-a)(x-3a)}$.

(15) 1.

(16) $\frac{2x^4+13a^2x^2-2a^3x-a^4}{(a^2-x^2)^2}$.

(17) $\frac{1+x+x^2}{1-x-x^4+x^5}$.

(18) $\frac{a+\sqrt{a^2-x^2}}{x}$.

(19) $\sqrt{1-x}$.

Exercises, No. 42. Page 63.

(1) 5 : 1. (2) 11 : 1. (3) $21\frac{4}{15}$.

Exercises, No. 43. Pages 69 to 71.

(3) $(a-b)^2 = a^2-2ab+b^2$, $(a-b)^3 = a^3-3a^2b+3ab^2-b^3$.

(7) $3(a^2+b^2+c^2)+2(ab+bc-ac)$.

(15) 93, $-68\frac{1}{2}$.

(18) A numerical magnitude. 7 : 15.

(24) $x = \frac{2y}{3}$. (25) 4 or $\frac{1}{2}$. (27) $(1-\frac{1}{m})^n$.

Exercises, No. 44. Page 71.

(1) The difference of the cubes of two quantities is exactly divisible by the difference of the quantities, and the quotient is the sum of the squares of the quantities together with their product.

(2) The fourth power of any quantity increased by four is equal to the product of two factors, one of which is the number two together with twice the quantity together with the square of the quantity, and the other the number two less twice the quantity plus the square of the quantity.

(3) and (4) are similar.

(5) The sum of the cubes of three quantities less three times their product is equal to the product of the sum of the quantities into a quantity which is made of the sum of their squares less the sum of their products two and two.

(6) The product of the sum of the squares of the first and third of four quantities into the sum of the squares of the second and fourth is equal to the square of the sum of the products of the two first and two last together with the square of the difference of the products of the extremes.

Exercises, No. 45. Page 71.

(1) $\dfrac{a^8-b^8}{a-b} = (a+b)(a^2+b^2)(a^4+b^4)$.

(2) If $a : b :: c : d$, $a+b : a-b :: c+d : c-d$.

(3) $a^2+b^2+c^2+2ab+2bc+2ca = (a+b+c)^2$.

(4) $(a+b+c)(a+b-c)(b+c-a)(c+a-b) = 2a^2b^2+2b^2c^2 + 2c^2a^2-(a^4+b^4+c^4)$.

Exercises, No. 46. Page 74.

(1) £10 10*s.* 6*d.* (3) £1 19*s.* $6\frac{1}{9}\frac{4}{1}\frac{3}{2}\frac{4}{5}d.$ (5) In 8 yrs.

(2) 5 %. (4) £475.

(6) $P =$ £71 5*s.*, $M =$ £88 7*s.*, $I =$ £17 2*s.*, $n = 8$, $r = \cdot 03$.

Given *Pnr*, *PMr*, *PMn*, *PIn*, *PIr*, *IMn*, *IMr*, *Mnr*, *Inr*, to find a fourth quantity and thence by (ii) the fifth.

Exercises, No. 47. Page 75.

(1) £17 17*s.* $1\frac{1}{2}d.$ $\frac{6}{7}q.$
(2) £14 7*s.* $0\frac{1}{2}d.$ $\frac{14}{17}q.$
(3) £692 6*s.* $10\frac{3}{4}d.$ $\frac{23}{27}q.$
(4) £70 18*s.* $1\frac{230}{633}d.$

Exercises, No. 48. Page 76.

(1) £34 15*s.* $5\frac{1}{4}d.$ (2) £69 5*s.* $4\frac{1}{2}d.$ (3) £114 15*s.* $0\frac{1}{2}d.$

Exercises, No. 49. Pages 77, 78.

(1) 13, 49. (2) 32, 187. (3) 9, 24. (4) -2, 49.
(5) -10, 10; 14, 8, 2, -4, -10.
(6) $a+13b$, $6a+48b$. (9) $10\frac{3}{4}$, 142. (12) 14, 28.
(7) 14, 105. (10) $35\frac{1}{2}$, 1900. (13) $-1\frac{7}{8}$, $-13\frac{3}{4}$.
(8) 50, 442. (11) $3\frac{7}{8}$, $60\frac{1}{8}$, (14) 5, $27\frac{1}{2}$.
(15) -20, -90; -10, -12, -14, -16, -18, -20.

Exercises, No. 50. Page 79.

(1) 81, 121.
(2) 256, 510.
(3) $\frac{3}{64}$, $5\frac{61}{64}$.
(4) $\frac{5}{27}$, $7\frac{11}{27}$.
(5) $-\frac{91}{72}$, $2\frac{41}{72}$.
(6) $4\frac{1}{2}$.
(10) $\frac{2}{3}$.
(7) $2\frac{2}{3}$.
(8) $\frac{1}{3}$.
(9) $\frac{23}{111}$.

Exercises, No. 51. Pages 86 to 88.

(1) 8.
(2) $1\frac{1}{5}$.
(3) $13b-4a+c-2$.
(4) 6.
(5) 24.
(6) 9.
(7) $-1\frac{7}{17}$.
(8) 12.
(9) 12.
(10) 3.
(11) $\frac{105}{684}$.
(12) 2.
(13) $139\frac{1}{2}$.
(14) 4.
(15) $16\frac{3}{8}$.
(16) 13.
(17) $23\frac{1}{4}$.
(18) $66\frac{2}{3}$.
(19) 72.
(20) $1\frac{1}{5}$.
(21) $1\frac{27}{73}$.
(22) $22\frac{190}{617}$.
(23) $3\frac{82}{191}$.
(24) $13\frac{69}{2563}$.
(25) $139\frac{1}{2}$.
(26) $\frac{3a-6}{4}$.
(27) $116\frac{148}{367}$.
(28) $\frac{6-3a}{6a-2b}$.

(29) $4\frac{23}{45}$.
(30) $6\frac{3}{8}$.
(31) $1\frac{29}{124}$.
(32) $\frac{af-cd}{ce-bf}$.
(33) 2·010424 ....
(34) 638·92283 ....
(35) $\frac{(h+g)\,bdf}{b(de+cf)+adf}$.
(36) $\frac{a^2e(c-d)}{(a^2+b^2)d}$.
(37) $\frac{ab}{a+b}$.
(38) 7.
(39) $-1\frac{1}{12}$.
(40) $3\frac{1}{2}$.

(41) $\frac{bd-ac \pm \sqrt{(a-b)(c-d)(a+d)(b+c)}}{a-b-c+d}$

(42) 4.
(43) $-1\frac{1}{6}$.
(44) $-\frac{25}{28}$.
(45) $\frac{5-e(d+b+ae-ce)}{d-b-ae-ce}$.

Exercises, No. 52. Pages 97 to 99.

I. (1) $2x-3$.
(2) $\frac{x}{2}-\frac{5}{2}$.
(3) $\frac{x}{60}$.
(4) $12(x-1)$.
(5) $\frac{1}{9}(3x+5)$.
(6) $\frac{1}{3}(x+40)$.
(7) $\frac{x}{10}$.
(8) $\frac{x}{33}$.
(9) $\frac{x}{3}$.

II. (1) $x-5$.
(2) $x+5$.
(3) 7, 5, 3, 1.
(4) $\frac{2x+10}{3}$.

III. (1) $\frac{x}{2}-\frac{1}{36}$.
(2) $37-3x$ yds.
(3) $\frac{2x}{5}-\frac{1}{45}$.
(4) $\frac{x}{12}, \frac{11x}{12}$.
(5) $\frac{x}{b}$ minutes.
(6) $\frac{3}{88}\frac{b+x}{a}$ min.
(7) $\frac{3}{88}\frac{b+x}{a-c}, \frac{3}{88}\frac{b+x}{a+c}$ min.

IV. (1) $\frac{x}{3}$.
(2) ax.
(3) $\frac{x}{a}$.
(4) $\frac{bx}{a}$.
(5) $\frac{1}{x}$ hrs.
(6) $\frac{b}{x}$ hrs.
(7) $\frac{ab}{x}$ hrs.
(8) $\frac{x}{60}$.
(9) $88x$.

(10) $\frac{\text{Length moved (in proper units)}}{\text{Time (in proper units)}}$. (11) 45 miles an hour.

(12) $\frac{3a}{88x}$, $\frac{3(a+b)}{88x}$, $\frac{3(a+b)}{176x}$. min.

V. (1) $\frac{ax}{100}$. (2) $\frac{ac}{b}$. (3) $\frac{21c}{20}$, $\frac{100+d}{100}c$.

(4) $\frac{c}{19}$, $\frac{dc}{100-d}$. (5) $\frac{a-x}{a}b$ galls.

Exercises, No. 53. Pages 99 to 107.

(1) 5*s.* 2*d.* (2) $3\frac{3}{4}$ miles per hour.

(3) $\frac{60c}{ab}(b-1)$, and 3 miles per hour.

(4) 8, 16. (9) £4 4*s.*

(5) $\frac{a}{m+1}$, $\frac{ma}{m+1}$, 32, 96. (10) £130, £520.

(11) £3225, £6450.

(6) 12, 48. (12) 10.

(7) £70, £245. (13) 30 men, 20 women.

(8) £1 5*s.* (14) £3.

(15) £12 1*s.* 6*d.*, £8 1*s.*, £6 0*s.* 9*d.*

(16) £297, £533 10*s.*, £704, £1197 18*s.*

(17) $\frac{3}{4}$*d.* each, 512.

(18) $43\frac{7}{11}$ minutes past eight.

(19) $10\frac{10}{11}$ minutes past eight.

(20) $x = \frac{60}{11}(m \pm 6)$, the negative sign being used when $m > 6$.
If $m = 4$, $x = 54\frac{6}{11}$ minutes, if $m = 7$, $x = 5\frac{5}{11}$ minutes.

(21) £16000, £4000.

(22) *A* two quarts, *B* one quart.

(23) £900 in Three per *Cents.*, £800 in Three and a *Quarter.*

(24) 60. (28) 2016, 1728. (32) £5400, £7200.

(25) 50. (29) 135, 297, 432. (33) 112 and 96.

(26) $977\frac{7}{9}$ yds. (30) 5, 8. (34) 355.

(27) $43\frac{1}{2}$ miles. (31) $30\frac{5}{24}$ inches. (35) 40, 50.

(36) 13. (39) 20. (42) £275.
(37) 16. (40) 10lbs. or $3\frac{7}{11}$lbs. (43) £1400.
(38) 8. (41) 80. (44) 12.
(45) $12\frac{1}{2}$, 14, $15\frac{1}{2}$. (46) 1*s.* 8*d.*
(47) $10\frac{10}{11}$ minutes, and $43\frac{7}{11}$ minutes past five.
(48) $13\frac{3}{4}$ miles. (50) 4 times as fast.
(49) Half a minute ($\frac{60}{121}$m.) to one. (51) 40 miles an hour.
(52) 22 miles, $29\frac{1}{11}$ miles from terminus.
(53) £6 13*s.* 4*d.* and £33 6*s.* 8*d.*
(54) $\frac{p-n}{m-n}a$, $\frac{m-p}{m-n}a$. (58) 150 half-crowns, 50 florins.
(55) £1 10*s.* $6\frac{6}{11}d.$ (59) £5 5*s.*
(56) 3*s.* 6*d.* (60) $6\frac{1}{4}$ per cent.
(57) 7 miles. (61) $1\frac{1}{2}$, $4\frac{1}{2}$ hrs.
(62) *A* £760, *B* £880, *C* £360, *D* £520.
(63) $65\frac{3}{4}°$.
(64) Interchange $\frac{1}{6}$ part of the heavier with $\frac{1}{6}$ of the lighter.
(65) $107\frac{1}{2}\frac{1}{7}$ per cent.
(66) £1000.
(67) $9\frac{1}{7}$ miles from Ely.
(68) 440.
(69) $\frac{9}{65}$ths of *A*'s contents.
(70) 90.
(71) $41\frac{3}{5}$ miles per hour.
(72) $\frac{10b}{g}-a-\frac{b}{10}$ galls.
(73) $\frac{9-6p+105r}{10r}$.
(74) 22 miles, 5 mil., 7 mil.
(75) 70, 90, 120 galls.
(76) Every 580 days nearly.
(77) Every 116 days nearly.
(78) 16 galls. milk, 4 galls. water.
(79) 38·14 kg. lead, 41·85 kg. cork. (80) $p-n : m-p$.
(81) 7 men, 9 days and 10 men, 7 days.
(82) $\frac{100np}{naq+nbp+mcp}$. (83) $a+b-\frac{ab}{l}$.
(84) $7\frac{1}{2}$ miles per hour.

Exercises, No. 54. Pages 111, 112.

(1) 5, −1. (4) 16, 35. (7) 6, 3. (10) 4, 3.
(2) $4\frac{3}{8}$, $2\frac{5}{8}$. (5) 7, 2. (8) 2, 6. (11) 3, 4.
(3) $7\frac{1}{2}$, $5\frac{1}{2}$. (6) 12, 6. (9) 4, 3. (12) 5, 2.

(13) 3, 2.
(14) $17\frac{1}{2}$, $115\frac{3}{5}$.
(15) $4\frac{1}{2}$, $8\frac{2}{7}$.
(16) $88\frac{3}{4}$, $17\frac{3}{4}$.
(17) 9, $123\frac{1}{2}$.
(18) $5\frac{1}{4}$, $2\frac{1}{10}$.
(19) -12, 50.
(20) $\frac{7}{6}$, $\frac{11}{3}$.
(21) 56, 23.
(22) $-1\frac{2}{9}$, $1\frac{5}{6}$.
(23) $-39{\cdot}812$, $58{\cdot}542$.
(24) 3, 5.
(25) $38\frac{6}{7}$, $-8\frac{6}{7}$.
(26) 3, 5.
(27) 7, 6.
(28) $\frac{m^2-n^2}{mb-na}$, $\frac{m^2-n^2}{ma-nb}$.
(29) $1\frac{17}{35}$, $5\frac{1}{4}$.
(30) 2, 9.
(31) $\frac{mp-nq}{pa}$, $\frac{mp-nq}{qa}$.

Exercises, No. 55. Pages 115 to 118.

(1) 69, 51.
(2) $\frac{1}{2}(a+b)$, $\frac{1}{2}(a-b)$.
(3) 16 at first: shares 3, 7.
(4) 125, 60.
(5) As 2 : 3.
(6) *A* £88, *B* £44.
(7) $\frac{32}{61}$.
(8) $\frac{2}{9}$.
(9) 2*s.* 6*d.* and 3*s.* 2*d.*
(10) 49.
(11) 6, 1.
(12) 24 lbs. and 32 lbs.
(13) £18 and £$40\frac{1}{2}$.
(14) £2 12*s.*, £52.
(15) £720 Three per Cents., £992 Three and a Quarter.
(16) 14 lbs. of hydrogen, 112 lbs. of oxygen, 5 lbs. of each left.
(17) 500 cubic feet oxygen, 1000 „ of hydrogen, } water;
273 „ of oxygen, 1027 „ of nitrogen, } air; and 100 cubic feet left.
(18) 2 gallons from the first, 12 gallons from the second.
(19) *A* 100 seconds fast, *B* 20 minutes slow; *A* gaining $16\frac{2}{3}$ seconds and *B* losing $3\frac{1}{3}$ minutes per hour.
(20) 169.
(21) Iron 720 lbs., stone 512 lbs.
(22) $A : C :: 5 : 6$; $B : C :: 9 : 11$; $A-B : A :: 1 : 1\frac{9}{11} : 100$.
(23) 26, 48.
(24) $\frac{7}{11}$.
(25) £3000, £5000.
(26) 1, 3, 5, 7, 9, 11.
(27) 28 fr., 16 fr.
(28) 1120 silver, 360 copper.

Exercises, No. 56. Page 120.

(1) ± 5. (4) ± 2. (7) ± 2.
(2) ± 1. (5) ± 6. (8) $\pm \sqrt{2}$.
(3) ± 6. (6) ± 1. (9) ± 3.

Exercises, No. 57. Pages 123 to 125.

(1) 5, 3.
(2) $-5, -7$.
(3) 3, -2.
(4) 5, -7.
(5) 4, -6.
(6) 11, -2.
(7) $2\frac{1}{2}, -2\frac{1}{3}$.
(8) $2\frac{2}{3}, -3\frac{1}{2}$.
(9) $\frac{5}{2}, \frac{1}{7}$.
(10) $1\frac{1}{2}, -2\frac{1}{2}$.
(11) 3, $-\frac{2}{3}$.
(12) $5\frac{1}{2}, 4\frac{1}{2}$.
(13) 2, $\frac{3}{5}$.
(14) 6, 5.
(15) $3\frac{3}{4}, 5\frac{17}{20}$.
(16) $6\frac{3}{7}, 3\frac{1}{4}$.
(17) $-25\frac{10}{39}, -52$.
(18) 1, -2.
(19) 6, -5.
(20) 14, -10.
(21) 2, $2\frac{1}{3}$.
(22) 2, $\frac{2}{3}$.
(23) 32, 7.
(24) 3, -4.
(25) 5, 1.
(26) 9, 6.
(27) 7, $-\frac{8}{3}$.
(28) 21, -5.
(29) 7, -1.
(30) 5, $-\frac{5}{2}$.
(31) $\frac{1}{3}a, -3a$.
(32) $\frac{2}{3}, \frac{3}{10}$.
(33) $13\frac{22}{31}$, 8.
(34) 5, $5\frac{2}{5}$.
(35) $\frac{1}{4}(9 \pm \sqrt{33})$.
(36) 5, 3.
(37) $2a, -a$.
(38) $2a, b-a$.
(39) $14\frac{3}{5}, \frac{72}{245}$.
(40) $7\frac{47}{112}$, 2.
(41) 6, $-9\frac{23}{47}$.
(42) a, b.
(43) $a, -\frac{c}{2}$.
(44) $-\frac{a \pm 1}{2a}$.
(45) 4 or $-2\frac{57}{87}$.
(46) $\frac{a}{6}(1 \pm \sqrt{13})$.
(47) -1.
(48) $\frac{2a-b}{ac}, -\frac{3a+2b}{bc}$.
(49) $\frac{2c+ad}{d(a+b)}, \frac{c}{d(a-b)}$.

Exercises, No. 58. Pages 125 to 127.

(1) 9.
(2) 72.
(3) 21.
(4) 15.
(5) 1 foot.
(6) £80.
(7) 42.
(8) 18.
(9) $56\frac{1}{4}$ min.
(10) 4, 2.
(11) 3, 12.
(12) 1024.

(13) 4 miles, 5 miles.
(14) 30, 40, 50 ft. square.
(15) 45 miles, 18 miles.
(16) 20, 10.
(17) $m+n$, $m-n$.
(18) 576.
(19) 9, 6.

Exercises, No. 59. Pages 130, 131.

(1) 11, 17; 17, 11.
(2) 13, 13.
(3) 21, 13; −13, −21.
(4) 23, 13; −13, −23.
(5) 19, 17; −19, −17.
(6) 23, 7; 7, 23.
(7) 27, 23; −23, −27.
(8) ± 18, ± 17.
(9) ± 33, ± 29.
(10) ± 26, ± 19.
(11) 37, 35.
(12) 31, 21.
(13) ± 23, ± 3.
(14) ± 27, ± 11.
(15) ± 41, ± 22.
(16) 14, 12; 12, 14.
(17) 19, 13; −13, −19.
(18) 17, 13; 13, 17.
(19) 15, 12; −12, −15.
(20) 19, 8; −8, −19.
(21) 11, 8; 8, 11.
(22) ± 13, ± 14.
(23) ± 8, ± 9.
(24) ± 30, $\pm 5\frac{1}{5}$.
(25) 3, 2; $-22\frac{13}{17}$, $19\frac{3}{17}$.
(26) $13, 3; 6\frac{7}{11}, -\frac{9}{11}$.

Exercises, No. 60. Pages 131, 132.

(1) 2, 10.
(2) 12, 19.
(3) 2, 11.
(4) 11, 17.
(5) 13, 15.
(6) 50, 15.
(7) 85, 76.
(8) 456, 342.
(9) $\frac{n\sqrt{b}}{\sqrt{m^2+n^2}}$, $\frac{m\sqrt{b}}{\sqrt{m^2+n^2}}$.
(10) 3, 6.
(11) 9, 11.
(12) 4, 12.
(13) 24, 15.
(14) 16, 4.

Exercises, No. 61. Pages 137, 138.

(1) $3x-2y$.
(2) $x+\frac{1}{2}$.
(3) $\frac{3x^4}{2}-\frac{2y^3}{3}$.
(4) $\frac{5}{2}ab+\frac{c^2}{3}$.
(5) $3x-5a-\frac{a^2}{2}$.
(6) $2a-3b+5c$.
(7) $3a-b+5c+d$.
(8) $3x-\frac{2}{x}$.
(9) $\frac{5a}{x}-\frac{2x}{a}$.

(10) $x^2-2+\frac{4}{x^2}$.

(11) $3y-5+\frac{1}{y}$.

(12) $a-3b-11c$.

(13) $x+\frac{2}{x}-\frac{3}{x^2}$.

(14) $1-2x+3x^2-4x^3$.

(15) $1-\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^6}{16}\ldots$.

(16) $1-\frac{x}{2}-\frac{5x^2}{8}-\frac{5x^3}{16}\ldots$.

(17) $1-\frac{x}{2}+\frac{3x^2}{8}+\frac{3x^3}{16}\ldots$.

Exercises, No. 62. Page 140.

(1) $2x-5$. (2) $2a-7x$. (3) $y-\frac{2}{y}$.

(4) $3z^2-2az+a^2$. (5) $2y-3+5z$.

(6) $1-\frac{2x^2}{3}-\frac{4x^4}{9}-\frac{40}{81}x^6-\ldots$.

(7) $1-\frac{x}{3}-\frac{4x^2}{9}-\frac{23x^3}{81}-\ldots$.

(8) $a-\frac{x^3}{3a^2}-\frac{x^6}{9a^5}-\frac{5x^9}{81a^8}-\ldots$.

(9) $1-x-\frac{x^2}{3}-\frac{x^3}{3}-\ldots$. (12) $x-1+\frac{2}{x}$.

(10) $5z-7a$. (13) $3a-2b+2c$.

(11) $1+3x-3x^2$. (14) $1-3x+2y$.

Exercises, No. 63. Pages 144, 145.

(1) 6 [Reading $a=2$]. (3) 9 [Reading $a=-4$].

(2) 7. (4) 7.

(9) a^2b^2, $a+6a^{\frac{1}{2}}b^{\frac{1}{2}}+9b$. (10) $a^{\frac{4}{3}}b^{\frac{4}{3}}$, a^2b^4.

(11) $a^{\frac{4}{3}}-6ab^{\frac{1}{3}}+11a^{\frac{2}{3}}b^{\frac{2}{3}}-6a^{\frac{1}{3}}b+b^{\frac{4}{3}}$.

(12) $8a-36a^{\frac{2}{3}}b^{\frac{1}{3}}+54a^{\frac{1}{3}}b^{\frac{2}{3}}-27b$.

(13) $a^{\frac{1}{2}}(1-\frac{x}{a}-\frac{x^2}{2a^2}-\frac{x^3}{2a^3}-\ldots)$.

(14) $\frac{1}{a^{\frac{1}{2}}}(1+\frac{2x}{3a}+\frac{8}{9}\frac{x^2}{a^2}+\frac{112}{81}\frac{x^3}{a^3}+\cdots)$.

(15) $1+\frac{3x}{2}+\frac{47}{8}x^2+\frac{315}{16}x^3+\ldots$.

(16) $2^{\frac{1}{3}}(1-\frac{5}{6}x+\frac{17}{36}x^2+\frac{635}{648}x^3-\ldots)$. (18) $6x+x^{\frac{5}{6}}-15x^{\frac{2}{3}}$.

(19) $10x^{\frac{13}{10}}+4x^{\frac{7}{6}}+6x+5x^{\frac{3}{5}}+15x^{\frac{7}{15}}+6x^{\frac{1}{5}}+3x^{\frac{3}{10}}+9x^{\frac{1}{6}}+2x^{-\frac{7}{15}}$.

(20) $2x^{-3}+4x^{-4}+5x^{-5}+2x^{-6}+10x^{-7}-3x^{-8}$.

(21) $x^{\frac{np+mq}{nq}}+x^{\frac{pu+qt}{qu}}-x^{\frac{ms+nr}{ns}}-x^{\frac{ru+st}{su}}$.

(22) $x^{\frac{3p}{q}}-3x^{\frac{2ps+qr}{qs}}+3x^{\frac{ps+2qr}{qs}}-x^{\frac{3r}{s}}$.

Exercises, No. 64. Pages 155, 156.

(1) 720.
(2) 24, *dabc, dacb, dbac, dbca, dcab, dcba.*
(3) 3628800.
(4) 120.
(5) 24.
(6) 75600.
(7) 9979200.
(8) 120.
(9) 120.
(10) 325.
(11) 65.
(12) 35.
(13) 105.
(14) 15.
(15) 64.
(16) 2160.
(17) 8.
(18) 5005.
(19) 5^{10}.
(20) n^m.
(21) 48.
(22) 15625.
(23) 5400.
(24) 20020.
(25) 6.
(26) $\frac{\lfloor mn}{\lfloor m \, \lfloor mn-n}$.

Exercises, No. 65. Page 162.

(1) $8-12x+6x^2-x^3$.

(2) $x^4-28x^3+294x^2-1372x+2401$.

(3) $32a^5-80a^4b+80a^3b^2-40a^2b^3+10ab^4-b^5$.

(4) $128a^7-3136a^6c+32928a^5c^2-192080a^4c^3+672280a^3c^4-1411788a^2c^5+1647086ac^6-823543$.

(5) $1-9x+36x^2-84x^3+126x^4-126x^5+84x^6-36x^7+9x^8-x^9$.

(6) $16-\frac{32}{y}+\frac{24}{y^2}-\frac{8}{y^3}+\frac{1}{y^4}$.

(7) $1-\frac{4}{x}+\frac{20}{3x^2}-\frac{160}{27x^3}+\frac{80}{27x^4}-\frac{64}{81x^5}+\frac{64}{729x^6}$.

(8) $625a^4-3500a^3b+7350a^2b^2-6860ab^3+2401b^4$.

(9) $1451520y^4z^4$.
(10) $-\frac{135y^3}{2z^3}$.
(11) $-24\frac{z^7}{125}$.
(12) $\frac{45}{4^7}x^6$.

Exercises, No. 66. Page 163.

(1) $9a^2-30a+31-\frac{10}{a}+\frac{1}{a^2}$.

(2) $1-4m+10m^2-20m^3+25m^4-24m^5+16m^6$.

(3) $p^3-9p^2q+33p^2r+27pq^2-363pr^2-198pqr-27q^3$
$+297q^2r-1089qr^2+1331r^3$.

(4) $1+6x+3x^2-28x^3-9x^4+54x^5-27x^6$.

(5) $-936z^3$. (6) $160z^3$.

Exercises, No. 67. Page 169.

(1) $a^{\frac{1}{2}}\left\{1-\frac{1}{4}\frac{x}{a}-\frac{3}{32}\frac{x^2}{a^2}-\frac{7}{128}\frac{x^3}{a^3}-\dots\right\}$.

(2) $1-\frac{6x}{7}-\frac{24}{49}x^2-\frac{176}{343}x^3-\dots$.

(3) $4-2y-\frac{1}{16}y-\frac{1}{96}y^2-\dots$.

(4) $1-\frac{x}{2}+\frac{3x^2}{8}+\frac{11x^3}{16}+\dots$.

(5) $1+6z+29z^2+126z^3+\dots$.

(6) $1+42z+1029z^2+19208z^3+\dots$.

(7) $-\frac{28875}{2\sqrt{2}}c^3$. (10) $-\frac{2^{15}.11.13.14}{3^9}x^9$.

(8) $\frac{2717}{1152}.(\frac{d}{b})^8\frac{1}{(2b)^{\frac{1}{8}}}$. (11) 3.3166248.

(9) $7340032x^6$. (12) 2.4662121.

Exercises, No. 68. Page 173.

(1) 11027. (2) 6315. (3) 3447.

(4) $.4201545\dots$. (5) $.4608$.

Exercises, No. 69. Page 175.

(1) $\frac{8}{2}$, $\frac{6}{5}$, 1, $\frac{6}{7}$. (2) $\frac{1}{7}$. (3) $\frac{1}{5}$. $\frac{1}{7}$.

Exercises, No. 70. Pages 178, 179.

(1) $\frac{1}{3}n(n+1)(n+2)$, 20.

(2) $\frac{1}{8}n(n+1)(4n-1)$, 44.

(3) $\frac{1}{3}n(4n^2+6n-1)$, 53.

(4) $\frac{1}{3}n(4n^2-1)$, 35.

(5) $\frac{1}{2}(6n^3-3n^2-n)$, 66.

(6) $n^2(2n^2-1)$, 153.

(7) $\frac{1}{4}n(n+1)(n+2)(n+3)$, 90. (9) $\frac{n(3n+5)}{4(n+1)(n+2)}$, $\frac{3}{4}$.

(8) $\frac{n}{2n+1}$, $\frac{1}{2}$. (10) $\frac{n}{3(4n+3)}$, $\frac{1}{15}$.

Exercises, No. 71. Page 180.

(1) 4, 3, 2. (4) 17, 22, 45. (7) 1, −3, 2.
(2) 6, −5, 2. (5) 6, −3, 2. (8) 1, 2, 3.
(3) 5, 2, −1. (6) 8, 7, 13. (9) 5, 7, 8.
(10) 67½, 135, 270. (11) 24 h. cr., 46 fl., 40s.
(12) £7 10*s*., £10, £12 10*s*.

Exercises, No. 72. Page 182.

The solutions which belong to the negative sign of the radical are enclosed in [ ].

(1) 9, [16]. (4) ± 6, $[\pm\sqrt{1\frac{2}{3}7}]$.
(2) 7, 5. (5) ± 5, $[\pm 4\sqrt{2}]$.
(3) 2, −5, $[-\frac{1}{2}(3\mp\sqrt{241})]$. (6) $\pm 4\sqrt{5}$.

Exercises, No. 73. Page 183.

(1) 4, 5, 6.
(2) $\sqrt{\frac{2}{3}}$, $\sqrt{\frac{3}{2}}$, $\sqrt{6}$.
(3) $\frac{2abc}{ac+ab-bc}$, $\frac{2abc}{ab+bc-ac}$, $\frac{2abc}{bc+ac-ab}$.
(4) $\pm\frac{a^2}{\sqrt{a^2+b^2+c^2}}$, $\pm\frac{b^2}{\sqrt{a^2+b^2+c^2}}$, $\pm\frac{c^2}{\sqrt{a^2+b^2+c^2}}$.
(5) 3, 2, 1; $\frac{1}{3}$, $\frac{1}{2}$, 1.
(6) $\frac{b^2+c^2-a^2}{2bc}$, $\frac{a^2+c^2-b^2}{2ac}$, $\frac{a^2+b^2-c^2}{2ab}$.

Exercises, No. 74. Page 184.

(1) 144. (2) 4. (3) $\frac{9\pm\sqrt{76}}{5}$. (4) 14 or $\frac{62}{25}$.
(5) 2, $-\frac{10}{13}$. (6) $\pm\frac{a^2-b^2}{2\sqrt{2(a^2+b^2)}}$. (7) $\frac{1}{4}(51\pm 30\sqrt{2})$, $\frac{17a}{4}$.

Exercises, No. 75. Page 190.

(1) $1+\sqrt{3}$. (4) $\dfrac{\sqrt{7}-\sqrt{2}}{\sqrt{3}}$. (7) $\frac{1}{3}(7+2\sqrt{10})$.

(2) $3-\sqrt{7}$. (5) $\sqrt{a-1}-1$. (8) 6.

(3) $\sqrt{7}+\sqrt{2}$. (6) $\frac{1}{3}(4+\sqrt{7})$. (9) 2.

(11) $3^{\frac{2}{3}}+3^{\frac{1}{3}}2^{\frac{1}{3}}+2^{\frac{2}{3}}$.

(12) $3^{\frac{4}{3}}-2^{\frac{1}{3}}3^{2}+2^{\frac{2}{3}}3^{\frac{2}{3}}-6+2^{\frac{4}{3}}3^{\frac{1}{3}}-2^{\frac{5}{3}}$.

Exercises, No. 76. Page 194.

(1) $x=2, 5, 8, 11$.
$y=7, 5, 3, 1$.

(2) $x=7, 12, 17, \ldots$.
$y=1, 3, 5, \ldots$.

(3) $x=5, 16, 27$.
$y=9, 6, 3$.

(4) $x=5, 12$.
$y=8, 3$.

(5) $x=3, 14, 25, \ldots$.
$y=1, 7, 13, \ldots$.

(6) $x=5, 14, 23$.
$y=19, 11, 3$.

(7) $x=2, 7, 12$.
$y=16, 9, 2$.

(8) $x=2, 7, 12, \ldots$.
$y=1, 14, 27, \ldots$.

(9) Half-crowns 0, 2, 4, 6, 8, 10.
Shillings 25, 20, 15, 10, 5, 0.

(10) Seven. Florins 0, 1, 2, 3, 4, 5, 6.
Sixpences 27, 23, 19, 15, 11, 7, 3.

(11) Four. (12) Twenty-three.

(13) Half-sovereigns 0, 1, 2, 3, 4, 5.
Half-crowns 21, 17, 13, 9, 5, 1.

(14) 51. (15) 61.

(16) Two; giving 6 shillings and receiving 1 half-crown: or, giving 11 shillings and receiving 3 half-crowns.

Exercises, No. 77. Page 199.

(1) +, −. Between −6 and −5: between 0 and +1.

(2) −, −. Between −16 and −15: between −1 and 0.

(3) +, −. Between −2 and −1: between 3 and 4.

(4) +, +. Between 0 and 1: between 10 and 11.

(5) ·8541019 ... (7) 2·0945514 ... (9) 2·3403099 ...
(6) 3·4843289 ... (8) 4·9348179 ... (10) 4·6684090 ...
(11) ·6684016 ... (12) 1·3027756 ...

Exercises, No. 78. Page 209.

(1) Quotients 2, 6, 2. Convergents $\frac{1}{0}, \frac{2}{1}, \frac{13}{6}, \frac{28}{13} = \frac{196}{91}$.

(2) Quotients 1, 2, 14, 1, 1, 35.
Convergents $\frac{1}{0}, \frac{1}{1}, \frac{3}{2}, \frac{43}{29}, \frac{46}{31}, \frac{89}{60}, \frac{3161}{2131}$.

(3) Quotients 6, 2, 1, 4, 3, 1, 3, 1, 7, 1, 2.
Convergents $\frac{1}{0}, \frac{6}{1}, \frac{13}{2}, \frac{19}{3}, \frac{89}{14}, \frac{286}{45}, \frac{375}{59}, \frac{1411}{222}, \frac{1786}{281}$,
$\frac{13913}{2189}, \frac{15699}{2470}, \frac{45311}{7129}$.

(4) Quotients 1, 1, 1, 1, 2, 1, 1, 1, 6, 1, 12, 2.
Convergents $\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{8}{13}, \frac{11}{18}, \frac{19}{31}, \frac{30}{49}, \frac{199}{325}, \frac{229}{374}$.

(5) $x = 2$, $y = 3$.

(6) $x = 2, 5, 8, 11, 14, \ldots$. $y = 5, 13, 21, 29, 37, \ldots$.

(7) $\frac{1}{0}, \frac{1}{1}, \frac{11}{10}, \frac{12}{11}, \frac{32}{32}, \frac{292}{267}$. 11 metres = 12 yds. nearly.

(8) $\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{18}{29}, \frac{23}{37}$. 5 m. = 8 kilom. (roughly).

(9) $\frac{9}{4}, \frac{38}{17}, \frac{161}{72}$.

(10) $\frac{21}{11}, \frac{44}{23}$.

Exercises, No. 79. Page 214.

(1) 1.
(2) 2.
(3) 3.
(4) 1.
(5) 0.
(6) $\bar{3}$.
(7) 3.
(8) $\bar{1}$.
(9) 2·90.
(10) $\bar{3}$·60.
(11) 6·70.
(12) $\bar{3}$·30.
(13) $\bar{1}$·85.
(14) $\bar{3}$·48.
(15) 1·40.
(16) 0·70.
(17) 0·34.
(18) 2000000.
(19) 600.
(20) ·7.
(21) ·8.
(22) ·00009.
(23) 3000.
(24) ·06.

Exercises, No. 80. Page 221.

(1) ·06692.
(2) 5487.
(3) 9·846.
(4) 2930.
(5) 133·3.
(6) ·00008704.
(7) ·03677.
(8) ·1470.
(9) 879600.
(10) ·0919.
(11) 1·475.
(12) 7367.
(13) ·9345.
(14) ·6852.
(15) ·4619.
(16) ·05048.
(17) £50 3*s.* 7*d.*
(18) £172 18*s.*
(19) £1208.
(20) £14000.
(21) £16665.
(22) £6363.
(23) £225 12*s.*
(24) £439.
(25) £862 4*s.*

Exercises, No. 81. Page 224.

(1) 1·0003191.
(2) 3·0011030.
(3) 1·8077232.
(4) ·8081686.
(5) 1·8078935.
(6) 1·0037710.
(7) 100·608.
(8) ·100783.
(9) 6·42012.
(10) 6·42061.
(11) 6·42669.
(12) 642·7074.

(13)

	1	2	3	4	5	6	7	8	9
436	44	87	131	174	218	262	305	349	392
438	44	88	131	175	219	263	307	350	394

(14) £129 9*s.*
(15) £34 2*s.* 7*d.*
(16) $31\frac{3}{20}$ years.
(17) £379.

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ADDITIONAL ERRATA.

Page 27, line 5, for $+3c$ read $-3c$.

Page 36, lines 10, 13, 14, for a^3 read a^2.

Page 184, equation (5), for $\sqrt{3x+1}$ read $\sqrt{3x+10}$.

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